Virginia Tech Regional Math Contest 1979*

Problem 1. Show that the right circular cylinder of volume V which has the least surface area is the one whose diameter is equal to its altitude. (The top and bottom are part of the surface.)

Problem 2. Let S be a set which is closed under the binary operation $\circ$, with the following properties:

(i) there is an element $e \in S$ such that $a \circ e = e \circ a = a$, for each $a \in S$,

(ii) $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$, for all $a, b, c, d \in S$.

Prove or disprove:

(a) $\circ$ is associative on S

(b) $\circ$ is commutative on S

Problem 3. Let A be an $n \times n$ nonsingular matrix with complex elements, and let $\bar{A}$ be its complex conjugate. Let $B = A\bar{A} + I$, where I is the $n \times n$ identity matrix.

Prove or disprove:

(a) $A^{-1}BA = \bar{B}$

(b) The determinant of $A\bar{A} + I$ is real.

Problem 4. Let $f(x)$ be continuously differentiable on $(0, \infty)$ and suppose $\lim_{x\to\infty} f'(x) = 0$.

Prove that $\lim_{x\to\infty} \frac{f(x)}{x} = 0$.

Problem 5. Show, for all positive integers $n = 1, 2, ...$, that 14 divides $3^{4n+2} + 5^{2n+1}$.

Problem 6. Suppose $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. Determine whether $\sum_{n=1}^{\infty} \frac{a_n}{S_n^2}$ converges, where $S_n = a_1 + a_2 + ... + a_n$.

Problem 7. Let S be a finite set of non-negative integers such that $|x - y| \in S$ for all $x, y \in S$.

(a) Give an example of such a set which contains ten elements.

(b) If A is a subset of S containing more than two-thirds of the elements of S, prove or disprove that every element of S is the sum or difference of two elements from A.

Problem 8. Let S be a finite set of polynomials in two variables, x and y. For n a positive integer, define $\Omega_n(S)$ to be the collection of all expressions $p_1p_2...p_k$, where $p_i \in S$ and $1 \le k \le n$. Let $d_n(S)$ indicate the maximum number of linearly independent polynomials in $\Omega_n(S)$. For example, $\Omega_2(\{x^2, y\}) = \{x^2, y, x^2y, x^4, y^2\}$ and $d_2(\{x^2, y\}) = 5$.

(a) Find $d_2(\{1, x, x + 1, y\})$.

(b) Find a closed formula in n for $d_n(\{1, x, y\})$.

(c) Calculate the least upper bound over all such sets of $\limsup_{n\to\infty} \frac{\log d_n(S)}{\log n}$.

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`

$\limsup_{n\to\infty} a_n = \lim_{n\to\infty}(\sup\{a_n, a_{n+1}, ...\})$, where sup means supremum or least upper bound.