Virginia Tech Regional Math Contest 1979: Solutions*

Problem 2. Let S be a set which is closed under the binary operation $\circ$, with the following properties:

(i) there is an element $e \in S$ such that $a \circ e = e \circ a = a$, for each $a \in S$,

(ii) $(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d)$, for all $a, b, c, d \in S$.

Prove or disprove:

(a) $\circ$ is associative on S

(b) $\circ$ is commutative on S

Solution 2. (a) Is $\circ$ associative on S?

Let's consider the given equation in (ii) and substitute $b = e$:

$$\begin{aligned} (a \circ b) \circ (c \circ d) &= (a \circ c) \circ (b \circ d) && \text{given} \\ (a \circ e) \circ (c \circ d) &= (a \circ c) \circ (e \circ d) && \text{substitute } b = e \\ a \circ (c \circ d) &= (a \circ c) \circ d && \text{given: } x \circ e = e \circ x = x \end{aligned}$$

This proves that $\boxed{\circ \text{ is associative on } S.}$ ■

Alternatively, we can set $c = e$ with similar results:

$$\begin{aligned} (a \circ b) \circ (c \circ d) &= (a \circ c) \circ (b \circ d) && \text{given} \\ (a \circ b) \circ (e \circ d) &= (a \circ e) \circ (b \circ d) && \text{substitute } c = e \\ (a \circ b) \circ d &= a \circ (b \circ d) && \text{given: } x \circ e = e \circ x = x \end{aligned}$$

This proves that $\boxed{\circ \text{ is associative on } S.}$ ■

However, we can't prove the same result if we set $a = e$ or $d = e$, why is that? Exercise for the reader.

(b) Is $\circ$ commutative on S?

For this case, let's take the same given equation in (ii) and consider the case where $a = e$ and $d = e$:

$$\begin{aligned} (a \circ b) \circ (c \circ d) &= (a \circ c) \circ (b \circ d) && \text{given} \\ (e \circ b) \circ (c \circ e) &= (e \circ c) \circ (b \circ e) && \text{substitute } a = e,\ d = e \\ b \circ c &= c \circ b && \text{given: } x \circ e = e \circ x = x \end{aligned}$$

This proves that $\boxed{\circ \text{ is commutative on } S.}$ ■

Problem 5. Show, for all positive integers $n = 1, 2, \ldots$, that 14 divides $3^{4n+2} + 5^{2n+1}$.

Solution 5. We can demonstrate this via induction. Recall that to prove something by induction, we have two steps:

*__Disclaimer:__ these are *unofficial* solutions; they were not reviewed for correctness. Please submit feedback at `https://github.com/mbrukman/math-contests` if you find errors. You can find official solutions to VTRMC contests at `https://personal.math.vt.edu/linnell/Vtregional/solutions.pdf`.

1. prove the base case
2. prove the inductive case, e.g., if something is true for n, it's also true for $n+1$

The above two steps let us claim that this is generally true for all n of interest.

Let's first consider the base case ($n = 1$):

$$3^{4n+2} + 5^{2n+1} = 3^{4+2} + 5^{2+1} = 729 + 125 = 854 = 14 \cdot 61$$

Thus, 14 divides that expression, so we have shown the base case.

Now let's consider the inductive case:

1. assume that 14 divides $3^{4n+2} + 5^{2n+1}$
2. prove that 14 divides $3^{4(n+1)+2} + 5^{2(n+1)+1}$

Note that we can assume that:

$$3^{4n+2} + 5^{2n+1} \equiv 0 \mod 14 \qquad (1)$$

Let's start with the second point:

$$\begin{aligned}
3^{4(n+1)+2} + 5^{2(n+1)+1} &\stackrel{?}{\equiv} 0 \mod 14 && \text{starting point} \\
3^{4n+4+2} + 5^{2n+2+1} &\stackrel{?}{\equiv} 0 \mod 14 && \text{expand} \\
3^4 \cdot 3^{4n+2} + 5^2 \cdot 5^{2n+1} &\stackrel{?}{\equiv} 0 \mod 14 && \text{factor to match starting point} \\
81 \cdot 3^{4n+2} + 25 \cdot 5^{2n+1} &\stackrel{?}{\equiv} 0 \mod 14 && \text{simplify} \\
56 \cdot 3^{4n+2} + 25 \cdot 3^{4n+2} + 25 \cdot 5^{2n+1} &\stackrel{?}{\equiv} 0 \mod 14 && \text{separate 81 into 25 and remainder} \\
56 \cdot 3^{4n+2} + 25 \cdot (3^{4n+2} + 5^{2n+1}) &\stackrel{?}{\equiv} 0 \mod 14 && \text{factor out 25} \\
56 \cdot 3^{4n+2} &\stackrel{?}{\equiv} 0 \mod 14 && \text{using eq. 1} \\
0 &\equiv 0 \mod 14 && \text{since } 56 \equiv 0 \mod 14
\end{aligned}$$

Above, we were able to first subtract the expression $25 \cdot (3^{4n+2} + 5^{2n+1})$ since it is divisible by 14 and then similarly, $56 \cdot 3^{4n+2}$ since it is also divisible by 14.

Since we have proven the inductive case, this completes the proof. ■