Virginia Tech Regional Math Contest 1980*

Problem 1. Let $*$ denote a binary operation on a set S with the property that

$$(w * x) * (y * z) = w * z \text{ for all } w, x, y, z \in S$$

Show:

(a) If $a * b = c$, then $c * c = c$.

(b) If $a * b = c$, then $a * x = c * x$ for all $x \in S$.

Problem 2. The sum of the first n terms of the sequence

$$1, (1+2), (1+2+2^2), ..., (1+2+...+2^{k-1}), ...$$

is of the form $2^{n+R} + Sn^2 + Tn + U$ for all $n > 0$. Find R, S, T and U.

Problem 3. Let $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \ ... \ \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ ... \ \cdot 2n}$.

(a) Prove that $\lim\limits_{n\to\infty} a_n$ exists.

(b) Show that $a_n = \dfrac{\left(1-\left(\frac{1}{2}\right)^2\right)\left(1-\left(\frac{1}{4}\right)^2\right)\cdots\left(1-\left(\frac{1}{2n}\right)^2\right)}{(2n+1)a_n}$.

(c) Find $\lim\limits_{n\to\infty} a_n$ and justify your answer.

Problem 4. Let $P(x)$ be any polynomial of degree at most 3. It can be shown that there are numbers x_1 and x_2 such that $\int_{-1}^{1} P(x)dx = P(x_1) + P(x_2)$, where x_1 and x_2 are independent of the polynomial P.

(a) Show that $x_1 = -x_2$.

(b) Find x_1 and x_2.

Problem 5. For $x > 0$, show that $e^x < (1+x)^{1+x}$.

Problem 6. Given the linear fractional transformation of x into $f_1(x) = \dfrac{2x-1}{x+1}$, define $f_{n+1}(x) = f_1(f_n(x))$ for $n = 1, 2, 3, ...$. It can be shown that $f_{35} = f_5$. Determine A, B, C, and D so that $f_{28}(x) = \dfrac{Ax+B}{Cx+D}$.

Problem 7. Let S be the set of all ordered pairs of integers (m,n) satisfying $m > 0$ and $n < 0$. Let $\langle$ be a partial ordering on S defined by the statement: $(m,n) \langle (m',n')$ if and only if $m \leq m'$ and $n \leq n'$. An example is $(5,-10) \langle (8,-2)$. Now let O be a completely ordered subset of S, i.e., if $(a,b) \in O$ and $(c,d) \in O$, then $(a,b) \langle (c,d)$ or $(c,d) \langle (a,b)$. Also let C denote the collection of all such completely ordered sets.

(a) Determine whether an arbitrary $O \in C$ is finite.

(b) Determine whether the cardinality $\|O\|$ of O is bounded for $O \in C$.

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`

(c) Determine whether $\|O\|$ can be countably infinite for any $O \in C$.

Problem 8. Let $z = x + iy$ be a complex number with x and y rational and with $|z| = 1$.

(a) Find two such complex numbers.

(b) Show that $\left|z^{2n} - 1\right| = 2\left|\sin n\theta\right|$, where $z = e^{i\theta}$.

(c) Show that $\left|z^{2n} - 1\right|$ is rational for every n.