Virginia Tech Regional Math Contest 1980: Solutions*

Problem 1. Let $*$ denote a binary operation on a set S with the property that

$$(w * x) * (y * z) = w * z \text{ for all } w, x, y, z \in S$$

Show:

(a) If $a * b = c$, then $c * c = c$.

(b) If $a * b = c$, then $a * x = c * x$ for all $x \in S$.

Solution 1. Part (a)

Use the given equation, substitute, and simplify:

$$\begin{aligned}
(w * x) * (y * z) &= w * z && \text{given} \\
(a * b) * (a * b) &= a * b && \text{substitute } w = a, x = b, y = a, z = b \\
c * c &= c && \text{simplify using } a * b = c
\end{aligned}$$

This proves that if $a * b = c$, then $c * c = c$. ■

Part (b)

Use the given equation with some substitutions and utilize the result from part 1:

$$\begin{aligned}
(w * x) * (y * z) &= w * z && \text{given} \\
(c * c) * (x * x) &= c * x && \text{substitute } w = c, x = c, y = x, z = x \\
c * (x * x) &= c * x && \text{simplify using } c * c = c \text{ from part (a)} \\
(a * b) * (x * x) &= c * x && \text{expand using } a * b = c \\
a * x &= c * x && \text{simplify using given equation}
\end{aligned}$$

This proves that if $a * b = c$, then $a * x = c * x$ for all $x \in S$. ■

Problem 2. The sum of the first n terms of the sequence

$$1, (1 + 2), (1 + 2 + 2^2), ..., (1 + 2 + ... + 2^{k-1}), ...$$

is of the form $2^{n+R} + Sn^2 + Tn + U$ for all $n > 0$. Find R, S, T and U.

Solution 2. The given sequence is already defined for us in the general case in the last element:

$$1, (1 + 2), (1 + 2 + 2^2), ..., (1 + 2 + ... + 2^{k-1}), ...$$

The i^{th} term of this sequence can thus be written as follows:

*****Disclaimer:** these are *unofficial* solutions; they were not reviewed for correctness. Please submit feedback at `https://github.com/mbrukman/math-contests` if you find errors. You can find official solutions to VTRMC contests at `https://personal.math.vt.edu/linnell/Vtregional/solutions.pdf`.

$$a_i = \sum_{i=1}^{n} 2^{i-1}$$

Then, the sum of the first n terms in the sequence is:

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \sum_{k=1}^{i} 2^{k-1}$$

However, we don't actually have to expand or simplify the double-summation. Since we have 4 unknowns, we can just compute the first 4 values of A_n, set them equal, and solve a system of 4 equations with 4 unknowns.

Let $A_n = \sum_{i=1}^{n} a_i$, and let's compute several of the initial values:

$$\begin{aligned}
A_1 &= a_1 & &= 1 & &= 1 \\
A_2 &= a_1 + a_2 & &= 1 + (1+2) & &= 4 \\
A_3 &= a_1 + a_2 + a_3 & &= 1 + (1+2) + (1+2+4) & &= 11 \\
A_4 &= a_1 + a_2 + a_3 + a_4 & &= 1 + (1+2) + (1+2+4) + (1+2+4+8) & &= 26
\end{aligned}$$

Additionally, per the given problem statement, we also know that

$$A_n = 2^{n+R} + Sn^2 + Tn + U$$

so let's set them equal for $n = 1, 2, 3, 4$:

$$\begin{aligned}
A_1 &= 1 & &= 2^{1+R} + S + T + U & &(\text{n} = 1) \quad (1) \\
A_2 &= 4 & &= 2^{2+R} + 4S + 2T + U & &(\text{n} = 2) \quad (2) \\
A_3 &= 11 & &= 2^{3+R} + 9S + 3T + U & &(\text{n} = 3) \quad (3) \\
A_4 &= 26 & &= 2^{4+R} + 16S + 4T + U & &(\text{n} = 4) \quad (4)
\end{aligned}$$

We now have 4 equations and 4 unknowns.

Reorganizing equation (1) to put R on one side, and all the other variables on the other side, we get:

$$\begin{aligned}
2^{1+R} + S + T + U &= 1 & &\text{from eq. 1} \\
2^{1+R} &= 1 - S - T - U & &\text{solve for } 2^{1+R} \ (5)
\end{aligned}$$

since 2^{1+R} appears in all other equations and R does not appear as a separate variable, we don't have to solve for R, and can substitute the result in (5) into the remaining equations (2), (3), and (4) to remove the use of variable R, leaving us with 3 equations and 3 unknowns (S, T, U).

Let's now take equation (2), separate out 2^{1+R} and substitute the result from (5):

$$\begin{aligned}
2 \cdot 2^{1+R} + 4S + 2T + U &= 4 & &\text{from eq. 2} \\
2 \cdot (1 - S - T - U) + 4S + 2T + U &= 4 & &\text{substitute from eq. 5} \\
2 - 2S - 2T - 2U + 4S + 2T + U &= 4 & &\text{expand} \\
2 + 2S - U &= 4 & &\text{simplify} \\
2S - 2 &= U & &\text{solve for } U \ (6)
\end{aligned}$$

Now let's take equation (3) and similarly substitute from both (5) and (6):

$$\begin{aligned}
2^2 \cdot 2^{1+R} + 9S + 3T + U &= 11 && \text{from eq. 3} \\
2^2 \cdot (1 - S - T - U) + 9S + 3T + U &= 11 && \text{substitute } 2^{1+R} \text{ from eq. 5} \\
4 - 4S - 4T - 4U + 9S + 3T + U &= 11 && \text{expand} \\
5S - T - 3U &= 7 && \text{simplify} \\
5S - T - 3(2S - 2) &= 7 && \text{substitute } U \text{ from eq. 6} \\
5S - T - 6S + 6 &= 7 && \text{expand} \\
-S - T &= 1 && \text{simplify} \\
-S - 1 &= T && \text{solve for } T \text{ (7)}
\end{aligned}$$

Finally, let's take equation (4) and substitute the results from (5), (6), and (7) to eliminate all other variables and solve for S:

$$\begin{aligned}
2^3 \cdot 2^{1+R} + 16S + 4T + U &= 26 && \text{from eq. 4} \\
2^3 \cdot (1 - S - T - U) + 16S + 4T + U &= 26 && \text{substitute } 2^{1+R} \text{ from eq. 5} \\
8 - 8S - 8T - 8U + 16S + 4T + U &= 26 && \text{expand} \\
8 + 8S - 4T - 7U &= 26 && \text{simplify} \\
8 + 8S - 4(-S - 1) - 7U &= 26 && \text{substitute } T \text{ from eq. 7} \\
8 + 8S - 4(-S - 1) - 7(2S - 2) &= 26 && \text{substitute } U \text{ from eq. 6} \\
8 + 8S + 4S + 4 - 14S + 14 &= 26 && \text{expand} \\
(8S + 4S - 14S) + (8 + 4 + 14) &= 26 && \text{combine like terms} \\
-2S + 26 &= 26 && \text{simplify} \\
-2S &= 0 && \text{simplify} \\
S &= 0 && \text{solve for } S \text{ (8)}
\end{aligned}$$

Now that we have $S = 0$, we can go back and substitute this into earlier equations to find the other unknowns. Let's start with eq. (7) and compute T:

$$T = -S - 1 = 0 - 1 = -1$$

Similarly, using eq. (6), we can compute U:

$$U = 2S - 2 = 0 - 2 = -2$$

and finally, to compute R, we can use eq. (5):

$$2^{1+R} = 1 - S - T - U = 1 - 0 - (-1) - (-2) = 4 = 2^2$$

Since $2^{1+R} = 2^2$, we know that $R = 1$. Hence, here are all the unknown values:

$$\boxed{R = 1; \quad S = 0; \quad T = -1; \quad U = -2} \qquad \blacksquare$$

Verification

Now that we've solved for R, S, T, and U, let's verify some of the A_n values to make sure it works out:

$$
\begin{aligned}
A_1 &= \boxed{1} & &= 2^{1+R} + S + T + U & &= 2^{1+1} + 0 - 1 - 2 & &= 4 - 3 & &= \boxed{1} \\
A_2 &= \boxed{4} & &= 2^{2+R} + 4S + 2T + U & &= 2^{2+1} + 0 - 2 - 2 & &= 8 - 4 & &= \boxed{4} \\
A_3 &= \boxed{11} & &= 2^{3+R} + 9S + 3T + U & &= 2^{3+1} + 0 - 3 - 2 & &= 16 - 5 & &= \boxed{11} \\
A_4 &= \boxed{26} & &= 2^{4+R} + 16S + 4T + U & &= 2^{4+1} + 0 - 4 - 2 & &= 32 - 6 & &= \boxed{26}
\end{aligned}
$$

Checks out!