Virginia Tech Regional Math Contest 1981*

Problem 1. The number $2^{48} - 1$ is exactly divisible by what two numbers between 60 and 70?

Problem 2. For which real numbers b does the function $f(x)$, defined by the conditions $f(0) = b$ and $f' = 2f - x$, satisfy $f(x) > 0$ for all $x \geq 0$?

Problem 3. Let A be non-zero square matrix with the property that $A^3 = 0$, where 0 is the zero matrix, but with A being otherwise arbitrary.

(a) Express $(I - A)^{-1}$ as a polynomial in A, where I is the identity matrix.

(b) Find a 3×3 matrix satisfying $B^2 \neq 0$, $B^3 = 0$.

Problem 4. Define $F(x)$ by $F(x) = \sum_{n=0}^{\infty} F_n x^n$ (wherever the series converges), where F_n is the nth Fibonacci number defined by $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}, n > 1$. Find an explicit closed form for $F(x)$.

Problem 5. Two elements A, B in a group G have the property $ABA^{-1}B = 1$, where 1 denotes the identity element in G.

(a) Show that $AB^2 = B^{-2}A$.

(b) Show that $AB^n = B^{-n}A$ for any integer n.

(c) Find u and v so that $(B^a A^b)(B^c A^d) = B^u A^v$.

Problem 6. With k a positive integer, prove that $(1 - k^{-2})^k \geq 1 - 1/k$.

Problem 7. Let $A = \{a_0, a_1, ...\}$ be a sequence of real numbers and define the sequence $A' = a'_0, a'_1, ...$ as follows for $n = 0, 1, ...$: $a'_{2n} = a_n$, $a'_{2n+1} = a_n + 1$. If $a_0 = 1$ and $A' = A$, find

(a) a_1, a_2, a_3 and a_4

(b) a_{1981}

(c) A simple general algorithm for evaluating a_n, for $n = 0, 1, ...$

Problem 8. Let

(i) $0 < a < 1$,

(ii) $0 < M_{k+1} < M_k$, for $k = 0, 1, ...$,

(iii) $\lim_{k\to\infty} M_k = 0$.

If $b_n = \sum_{k=0}^{\infty} a^{n-k} M_k$, prove that $\lim_{n\to\infty} b_n = 0$.

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`