Virginia Tech Regional Math Contest 1982*

Problem 1. What is the remainder when $X^{1982} + 1$ is divided by $X - 1$? Verify your answer.

Problem 2. A box contains marbles, each of which is red, white or blue. The number of blue marbles is a least half the number of white marbles and at most one-third the number of red marbles. The number which are white or blue is at least 55. Find the minimum possible number of red marbles.

Problem 3.

Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors such that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is linearly dependent. Show that

$$\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix} = 0$$

Problem 4. Prove that $t^{n-1} + t^{1-n} < t^n + t^{-n}$ when $t \neq 1$, $t > 0$ and n is a positive integer.

Problem 5. When asked to state the Maclaurin Series, a student writes (incorrectly)

$$(*) \quad f(x) = f(x) + xf'(x) + \frac{x^2}{2!}f''(x) + \frac{x^3}{3!}f'''(x) + ...$$

(a) State Maclaurin's Series for $f(x)$ correctly.

(b) Replace the left-hand side of $(*)$ by a simple closed form expression in f in such a way that the statement becomes valid (in general).

Problem 6. Let S be a set of positive integers and let E be the operation on the set of subsets of S defined by $EA = \{x \in A \mid x \text{ is even}\}$, where $A \subseteq S$. Let CA denote the complement of A in S. $ECEA$ will denote $E(C(EA))$, etc.

(a) Show that $ECECEA = EA$.

(b) Find the maximum number of distinct subsets of S that can be generated by applying the operations E and C to a subset A of S an arbitrary number of times in any order.

Problem 7. Let $p(x) = ax^2 + bx + c$, where a, b and c are integers, with the property that $1 < p(1) < p(p(1)) < p(p(p(1)))$. Show that $a \geq 0$.

Problem 8. For $n \geq 2$, let $S_n = \displaystyle\sum_{k=n}^{\infty} \frac{1}{k^2}$.

(a) Prove or disprove that $\dfrac{1}{n} < S_n < \dfrac{1}{n-1}$.

(b) Prove or disprove that $S_n < \dfrac{1}{n - \frac{3}{4}}$.

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`