Virginia Tech Regional Math Contest 1983*

Problem 1. In the expansion of $(a+b)^n$, where n is a natural number, there are $n+1$ dissimilar terms. Find the number of dissimilar terms in the expansion of $(a+b+c)^n$.

Problem 2. A positive integer N (in base 10) is called *special* if the operation C of replacing each digit d of N by its nine's-complement $9-d$, followed by the operation R of reversing the order of the digits, results in the original number. (For example, 3456 is a special number because $R[(C3456)] = 3456$.) Find the sum of all special positive integers less than one million which do not end in zero or nine.

Problem 3. Let a triangle have vertices at $O(0,0)$, $A(a,0)$, and $B(b,c)$ in the (x,y)-plane.

(a) Find the coordinates of a point $P(x,y)$ in the exterior of $\triangle OAB$ satisfying area$(\triangle OAP)$ = area$(\triangle OBP)$ = area$(\triangle ABP)$.

(b) Find the coordinates of a point $Q(x,y)$ in the interior of $\triangle OAB$ satisfying area$(\triangle OAQ)$ = area$(\triangle OBQ)$ = area$(\triangle ABQ)$.

Problem 4. A finite set of roads connect n towns $T_1, T_2, ..., T_n$ where $n \geq 2$. We say that towns T_i and T_j $(i \neq j)$ are directly connected if there is a road segment connecting T_i and T_j which does not pass through any other town. Let $f(T_k)$ be the number of other towns directly connected to T_k. Prove that f is not one-to-one.

Problem 5. Find the function $f(x)$ such that for all $L \geq 0$, the area under the graph of $y = f(x)$ and above the x-axis from $x = 0$ to $x = L$ equals the arc length of the graph from $x = 0$ to $x = L$. Hint: recall that $\dfrac{d}{dx}\cosh^{-1} x = \dfrac{1}{\sqrt{x^2-1}}$.

Problem 6. Let $f(x) = 1/x$ and $g(x) = 1-x$ for $x \in (0,1)$. List all distinct functions that can be written in the form $f \circ g \circ f \circ g \circ \cdots \circ f \circ g \circ f$ where $\circ$ represents composition. Write each function in the form $\dfrac{ax+b}{cx+d}$ and prove that your list is exhaustive.

Problem 7. If a and b are real, prove that $x^4 + ax + b = 0$ cannot have only real roots.

Problem 8. A sequence f_n is generated by the recurrence formula

$$f_{n+1} = \frac{f_n f_{n-1} + 1}{f_{n-2}}$$

for $n = 2, 3, 4, \ldots$, with $f_0 = f_1 = f_2 = 1$. Prove that f_n is integer-valued for all integers $n \geq 0$.

*Source: https://personal.math.vt.edu/plinnell/Vtregional/