Virginia Tech Regional Math Contest 1985*

Problem 1. Prove that $\sqrt{ab} \leq (a+b)/2$ where a and b are positive real numbers.

Problem 2. Find the remainder r, $1 \leq r \leq 13$, when 2^{1985} is divided by 13.

Problem 3. Find real numbers c_1 and c_2 so that $I + c_1 M + c_2 M^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, where $M = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ and I is the identity matrix.

Problem 4. Consider an infinite sequence $\{c_k\}_{k=0}^{\infty}$ of circles. The largest, C_0, is centered at $(1,1)$ and is tangent to both the x and y-axes. Each smaller circle C_n is centered on the line through $(1,1)$ and $(2,0)$ and is tangent to the next larger circle C_{n1} and to the x-axis. Denote the diameter of C_n by d_n for $n = 0, 1, 2, ...$

Find:

(a) d_1

(b) $\sum_{n=0}^{\infty} d_n$

Problem 5. Find the function $f = f(x)$, defined and continuous on $\mathbb{R}^+ = \{x \mid 0 \leq x < \infty\}$, that satisfies $f(x+1) = f(x) + x$ on $\mathbb{R}^+$ and $f(1) = 0$.

Problem 6.

(a) Find an expression for $3/5$ as a finite sum of distinct reciprocals of positive integers. (For example: $2/7 = 1/7 + 1/8 + 1/56$.)

(b) Prove that any positive rational number can be so expressed.

Problem 7. Let $f = f(x)$ be a real function of a real variable which has continuous third derivative and which satisfies, for a given c and all real x, $x \neq c$,

$$\frac{f(x)f(c)}{x-c} = \frac{f'(x) + f'(c)}{2}.$$

Show that $f(x) = \dfrac{f'(x - f'(c))}{xc}$.

Problem 8. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, where the coefficients a_i are real. Prove that $p(x) = 0$ has at least one root in the interval $0 \leq x \leq 1$ if $a_0 + a_1/2 + \cdots + a_n/(n+1) = 0$.

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`