Virginia Tech Regional Math Contest 1986*

Problem 1. Let $x_1 = 1$, $x_2 = 3$, and $x_{n+1} = \dfrac{1}{n+1}\sum_{i=1}^{n} x_i$ for $n = 2, 3, ...$

Find $\lim_{n\to\infty} x_n$ and give a proof of your answer.

Problem 2. Given that $a > 0$ and $c > 0$, find a necessary and sufficient condition on b so that $ax^2+bx+c > 0$ for all $x > 0$.

Problem 3. Express $\sinh 3x$ as a polynomial in $\sinh x$. As an example, the identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ can be expressed as a polynomial in $\cos x$. Recall that sinh denotes the hyperbolic sine defined by:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Problem 4. Find the quadratic polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ such that:

$$\int_0^1 t^n p(t)dt = n \quad \text{for } n = 0, 1, 2.$$

Problem 5. Verify that, for $f(x) = x + 1$,

$$\lim_{r\to 0^+}\left(\int_0^1 (f(x))^r\, dx\right)^{\frac{1}{r}} = e^{\int_0^1 \ln f(x) dx}$$

Problem 6. Sets A and B are defined by $A = \{1, 2, ..., n\}$ and $B = \{1, 2, 3\}$. Determine the number of distinct functions from A onto B.

Recall that a function $f : A \to B$ is "onto" if for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Problem 7. A function f from the positive integers to the positive integers has the properties:

- $f(1) = 1$,
- $f(n) = 2$ if $n \geq 100$,
- $f(n) = f\left(\frac{n}{2}\right)$ if n is even and $n < 100$,
- $f(n) = f(n^2 + 7)$ if n is odd and $n > 1$.

(a) Find all positive integers n for which the stated properties require that $f(n) = 1$.

(b) Find all positive integers n for which the stated properties do not determine $f(n)$.

Problem 8. Find all pairs (M, N) of positive integers, $M < N$, such that:

$$\sum_{j=M}^{N} \frac{1}{j(j+1)} = \frac{1}{10}$$

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`