

Virginia Tech Regional Math Contest 1992*

Problem 1. Find the inflection point of the graph of $F(x) = \int_0^{x^3} e^{t^2} dt$, for $x \in \mathbb{R}$.

Problem 2. Assume that $x_1 > y_1 > 0$ and $y_2 > x_2 > 0$. Find a formula for the shortest length l of a planar path that goes from (x_1, y_1) to (x_2, y_2) and that touches both the x -axis and the y -axis. Justify your answer.

Problem 3. Let $f_n(x)$ be defined recursively by

$$f_0(x) = x, \quad f_1(x) = f(x), \quad f_{n+1}(x) = f(f_n(x)), \quad \text{for } n \geq 0,$$

where $f(x) = 1 + \sin(x - 1)$.

(a) Show that there is a unique point x_0 such that $f_2(x_0) = x_0$.

(b) Find $\sum_{n=0}^{\infty} \frac{f_n(x_0)}{3^n}$ with the above x_0 .

Problem 4. Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $t_1 = 1$ and $t_{n+1}^2 = 1 + t_n$, for $n \geq 1$. Show that t_n is increasing in n and find $\lim_{n \rightarrow \infty} t_n$.

Problem 5. Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$. Find A^{100} . You have to find all four entries.

Problem 6. Let $p(x)$ be the polynomial $p(x) = x^3 + ax^2 + bx + c$. Show that if $p(r) = 0$, then

$$\frac{p(x)}{x-r} - 2 \frac{p(x+1)}{x+1-r} + \frac{p(x+2)}{x+2-r} = 2$$

for all x except $x = r, r - 1$ and $r - 2$.

Problem 7. Find $\lim_{n \rightarrow \infty} \frac{2 \log 2 + 3 \log 3 + \cdots + n \log n}{n^2 \log n}$.

Problem 8. Some goblins, N in number, are standing in a row while “trick-or-treat”-ing. Each goblin is at all times either 2' tall or 3' tall, but can change spontaneously from one of these two heights to the other at will. While lined up in such a row, a goblin is called a **Local Giant Goblin** (LGG) if he/she/it is not standing beside a taller goblin. Let $G(N)$ be the total of all occurrences of LGG's as the row of N goblins transmogrifies through all possible distinct configurations, where height is the only distinguishing characteristic. As an example, with $N = 2$, the distinct configurations are $\hat{2}\hat{2}$, $2\hat{3}$, $\hat{3}2$, $\hat{3}\hat{3}$, where a cap indicates an LGG. Thus $G(2) = 6$.

(a) Find $G(3)$ and $G(4)$.

(b) Find, with proof, the general formula for $G(N)$, $N = 1, 2, 3, \dots$

*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>