

Virginia Tech Regional Math Contest 1994*

Problem 1. Evaluate $\int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2} dz dy dx$.

Problem 2. Let f be continuous real function, strictly increasing in an interval $[0, a]$ such that $f(0) = 0$. Let g be the inverse of f , i.e., $g(f(x)) = x$ for all x in $[0, a]$. Show that for $0 \leq x \leq a$, $0 \leq y \leq f(a)$, we have

$$xy \leq \int_0^x f(t) dt + \int_0^y g(t) dt.$$

Problem 3. Find all continuously differentiable solutions $f(x)$ for

$$f(x)^2 = \int_0^x (f(t)^2 - f(t)^4 + (f'(t))^2) dt + 100$$

where $f(0)^2 = 100$.

Problem 4. Consider the polynomial equation $ax^4 + bx^3 + x^2 + bx + a = 0$, where a and b are real numbers, and $a > 1/2$. Find the maximum possible value of $a + b$ for which there is at least one positive real root of the above equation.

Problem 5. Let $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$ be a function which satisfies $f(0, 0) = 1$ and

$$f(m, n) + f(m + 1, n) + f(m, n + 1) + f(m + 1, n + 1) = 0$$

for all $m, n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and all real numbers, respectively). Prove that $|f(m, n)| \geq 1/3$, for infinitely many pairs of integers (m, n) .

Problem 6. Let A be an $n \times n$ matrix and let α be an n -dimensional vector such that $A\alpha = \alpha$. Suppose that all the entries of A and α are positive real numbers. Prove that α is the only linearly independent eigenvector of A corresponding to the eigenvalue 1. Hint: if β is another eigenvector, consider the minimum of $\alpha_i/|\beta_i|$, $i = 1, \dots, n$, where the α_i 's and β_i 's are the components of α and β , respectively.

Problem 7. Define $f(1) = 1$ and $f(n + 1) = 2\sqrt{f(n)^2 + n}$ for $n \geq 1$. If $N \geq 1$ is an integer, find $\sum_{n=1}^N \frac{1}{f(n)^2}$.

Problem 8. Let a sequence $\{x_n\}_{n=0}^\infty$ of rational numbers be defined by $x_0 = 10$, $x_1 = 29$ and $x_{n+2} = \frac{19x_{n+1}}{94x_n}$ for $n \geq 0$. Find $\sum_{n=0}^\infty \frac{x_{6n}}{2^n}$.

*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>