

Virginia Tech Regional Math Contest 1998*

Problem 1. Let $f(x, y) = \ln(1 - x^2 - y^2) - \frac{1}{(y - x)^2}$ with domain $D = \{(x, y) | x \neq y, x^2 + y^2 < 1\}$. Find the maximum value M of $f(x, y)$ over D . You have to show that $M \geq f(x, y)$ for every $(x, y) \in D$. Here $\ln(\cdot)$ is the natural logarithm function.

Problem 2. The radius of the base of a right circular cone is 1. The vertex of the cone is V , and P is a point on the circumference of the base. The length of PV is 6 and the midpoint of PV is M . A piece of string is attached to M and wound tightly twice round the cone finishing at P . What is the length of the string?

Problem 3. Find the volume of the region which is common to the interiors of the three circular cylinders $y^2 + z^2 = 1$, $z^2 + x^2 = 1$ and $x^2 + y^2 = 1$.

Problem 4. Let ABC be a triangle and let P be a point on AB . Suppose $\angle BAC = 70^\circ$, $\angle APC = 60^\circ$, $AC = \sqrt{3}$ and $PB = 1$. Prove that ABC is an isosceles triangle.

Problem 5. Let a_n be sequence of positive numbers ($n = 1, 2, \dots$, $a_n \neq 0$ for all n), and let $b_n = (a_1 + \dots + a_n)/n$, the average of the first n numbers of the sequence. Suppose $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is a convergent series.

Prove that $\sum_{n=1}^{\infty} \frac{1}{b_n}$ is also a convergent series.

Problem 6. Ten cats are sitting on ten fence posts, numbered 1 through 10 in clockwise order and encircling a pumpkin patch. The cat on post #1 is white and the other nine cats are black. At 9:45 p.m. the cats begin a strange sort of dance. They jump from post to post according to the following two rules, applied in alternation at one second intervals. Rule 1: each cat jumps clockwise to the next post. Rule 2: all pairs of cats whose post numbers have a product that is 1 greater than a multiple of 11 exchange places. At 10 p.m., just as the Great Pumpkin rises out of the pumpkin patch, the dance stops abruptly and the cats look on in awe. If the first jump takes place according to Rule 1 at 9:45:01, and the last jump occurs at 10:00:00, on which post is the white cat sitting when the dance stops? (The first few jumps take the white cat from post 1 to posts 2, 6, 7, ...)

*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>