Virginia Tech Regional Math Contest 1999*

Problem 1. Let G be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, satisfying the following properties:

(a) $f(x) = f(x+1)$ for all x,

(b) $\displaystyle\int_0^1 f(x)dx = 1999$.

Show that there is a number α such that $\alpha = \displaystyle\int_0^1 \int_0^x f(x+y)dydx$ for all $f \in G$.

Problem 2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and satisfies both of the following properties:

(a) $f(1) = 2$,

(b) If α, β are real numbers satisfying $\alpha^2 + \beta^2 = 1$, then $f(\alpha x)f(\beta x) = f(x)$ for all x.

Find $f(x)$. Guesswork will not be accepted.

Problem 3. Let ε, M be positive real numbers, and let $A_1, A_2, \ldots$ be a sequence of matrices such that for all n,

(a) A_n is an $n \times n$ matrix with integer entries,

(b) The sum of the absolute values of the entries in each row of A_n is at most M.

If δ is a positive real number, let $e_n(\delta)$ denote the number of nonzero eigenvalues of A_n which have absolute value less that δ. (Some eigenvalues can be complex numbers.) Prove that one can choose $\delta > 0$ so that $e_n(\delta)/n < \varepsilon$ for all n.

Problem 4. A rectangular box has sides of length 3, 4, 5. Find the volume of the region consisting of all points that are within distance 1 of at least one of the sides.

Problem 5. Let $f : \mathbb{R}\_+ \to \mathbb{R}\_+$ be a function from the set of positive real numbers to the same set satisfying $f(f(x)) = x$ for all positive x. Suppose that f is infinitely differentiable for all positive x, and that $f(a) \neq a$ for some positive a. Prove that $\lim_{x\to\infty} f(x) = 0$.

Problem 6. A set S of distinct positive integers has property **ND** if no element x of S divides the sum of the integers in any subset of $S \setminus \{x\}$. Here $S \setminus \{x\}$ means the set that remains after x is removed from S.

(a) Find the smallest positive integer n such that $\{3, 4, n\}$ has property **ND**.

(b) If n is the number found in (i), prove that no set S with property **ND** has $\{3, 4, n\}$ as a proper subset.

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`