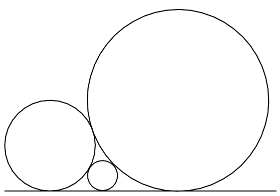
Virginia Tech Regional Math Contest 2001*

Problem 1. Three infinitely long circular cylinders each with unit radius have their axes along the x, y and z-axes. Determine the volume of the region common to all three cylinders. (Thus one needs the volume common to $\{y^2 + z^2 \leq 1\}$, $\{z^2 + x^2 \leq 1\}$, $\{x^2 + y^2 \leq 1\}$.)

Problem 2. Two circles with radii 1 and 2 are placed so that they are tangent to each other and a straight line. A third circle is nestled between them so that it is tangent to the first two circles and the line. Find the radius of the third circle.



Problem 3. For each positive integer n, let S_n denote the total number of squares in an $n \times n$ square grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chess board (i.e., determine S_8).

Problem 4. Let a_n be the nth positive integer k such that the greatest integer not exceeding $\sqrt{k}$ divides k, so the first few terms of $\{a_n\}$ are $\{1, 2, 3, 4, 6, 8, 9, 12, \dots\}$. Find a_{10000} and give reasons to substantiate your answer.

Problem 5. Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$.

That is, determine the real numbers x for which the above power series converges; you must determine correctly whether the series is convergent at the end points of the interval.

Problem 6. Find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(f(x)) = \dfrac{3x+1}{x+3}$ for all positive real numbers x (here $\mathbb{R}^+$ denotes the positive (nonzero) real numbers).

Problem 7. Let G denote a set of invertible 2×2 matrices (matrices with complex numbers as entries and determinant nonzero) with the property that if a, b are in G, then so are ab and a^{-1}. Suppose there exists a function $f : G \to \mathbb{R}$ with the property that either $f(ga) > f(a)$ or $f(g^{-1}a) > f(a)$ for all a, g in G with $g \neq I$ (here I denotes the identity matrix, $\mathbb{R}$ denotes the real numbers, and the inequality signs are strict inequality). Prove that given finite nonempty subsets A, B of G, there is a matrix in G which can be written in exactly one way in the form xy with x in A and y in B.

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`