

## Virginia Tech Regional Math Contest 2003\*

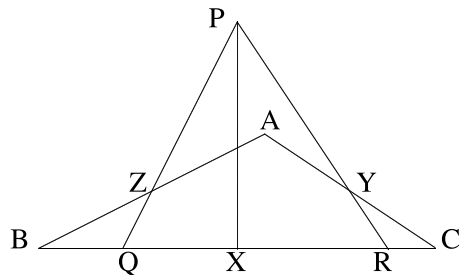
**Problem 1.** An investor buys stock worth 20 and holds it for  $n$  business days. Each day he has an equal chance of either gaining 10 or losing 10; he is deemed to have lost all his money, because he must have been involved with insider trading. Find a (simple) formula, with proof, of the amount of money he will have on average at the end of the  $n$  days.

**Problem 2.** Find  $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \dots$  for  $|x| < 1$ .

**Problem 3.** Determine all invertible 2 by 2 matrices  $A$  with complex numbers as entries satisfying  $A = A^{-1} = A^{\top}$ , where  $A^{\top}$  denotes the transpose of  $A$ .

**Problem 4.** It is known that  $2 \cos^3 \frac{\pi}{7} - \cos^2 \frac{\pi}{7} - \cos \frac{\pi}{7}$  is a rational number. Write this rational number in the form  $p/q$ , where  $p$  and  $q$  are integers with  $q$  positive.

**Problem 5.** In the diagram below,  $X$  is the midpoint of  $BC$ ,  $Y$  is the midpoint of  $AC$ , and  $Z$  is the midpoint of  $AB$ . Also  $\angle ABC + \angle PQC = \angle ACB + \angle PRB = 90^\circ$ . Prove that  $\angle PXC = 90^\circ$ .



**Problem 6.** Let  $f : [0, 1] \rightarrow [0, 1]$  be a continuous function such that  $f(f(f(x))) = x$  for all  $x \in [0, 1]$ . Prove that  $f(x) = x$  for all  $x \in [0, 1]$ . Here  $[0, 1]$  denotes the closed interval of all real numbers between 0 and 1, including 0 and 1.

**Problem 7.** Let  $T$  be a solid tetrahedron whose edges all have length 1. Determine the volume of the region consisting of points which are at distance at most 1 from some point in  $T$  (your answer should involve  $\sqrt{2}, \sqrt{3}, \pi$ ).

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\*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>