Virginia Tech Regional Math Contest 2006*

Problem 1. Find, and give a proof of your answer, all positive integers n such that neither n nor n^2 contain a 1 when written in base 3.

Problem 2. Let $S(n)$ denote the number of sequences of length n formed by the three letters A, B, C with the restriction that the C's (if any) all occur in a single block immediately following the first B (if any). For example ABCCAA, AAABAA, and ABCCCC are counted in, but ACACCB and CAAAAA are not. Derive a simple formula for $S(n)$ and use it to calculate $S(10)$.

Problem 3. Recall that the Fibonacci numbers $F(n)$ are defined by $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for $n \geq 2$. Determine the last digit of $F(2006)$ (e.g., the last digit of 2006 is 6).

Problem 4. We want to find functions $p(t), q(t), f(t)$ such that

(a) p and q are continuous functions on the open interval $(0, \pi)$.

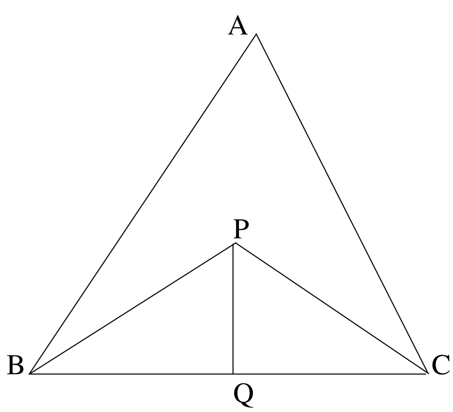
(b) f is an infinitely differentiable nonzero function on the whole real line $(-\infty, \infty)$ such that $f(0) = f'(0) = f''(0)$.

(c) $y = \sin t$ and $y = f(t)$ are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on $(0, \pi)$.

Is this possible? Either prove this is not possible, or show this is possible by providing an explicit example of such f, p, q.

Problem 5. Let $\{a_n\}$ be a monotonic decreasing sequence of positive real numbers with limit 0 (so $a_1 \geq a_2 \geq \cdots \geq 0$). Let $\{b_n\}$ be a rearrangement of the sequence such that for every non-negative integer m, the terms $b_{3m+1}, b_{3m+2}, b_{3m+3}$ are a rearrangement of the terms $a_{3m+1}, a_{3m+2}, a_{3m+3}$ (thus, for example, the first 6 terms of the sequence $\{b_n\}$ could be $a_3, a_2, a_1, a_4, a_6, a_5$). Prove or give a counterexample to the following statement: the series $\sum_{n=1}^{\infty} (-1)^n b_n$ is convergent.

Problem 6. In the diagram below BP bisects $\angle ABC$, CP bisects $\angle BCA$, and PQ is perpendicular to BC. If $BQ \cdot QC = 2PQ^2$, prove that $AB + AC = 3BC$.



Problem 7. Three spheres each of unit radius have centers P, Q, R with the property that the center of each sphere lies on the surface of the other two spheres. Let C denote the cylinder with cross-section PQR (the triangular lamina with vertices P, Q, R) and axis perpendicular to PQR. Let M denote the space which

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`

is common to the three spheres and the cylinder C, and suppose the mass density of M at a given point is the distance of the point from PQR. Determine the mass of M.