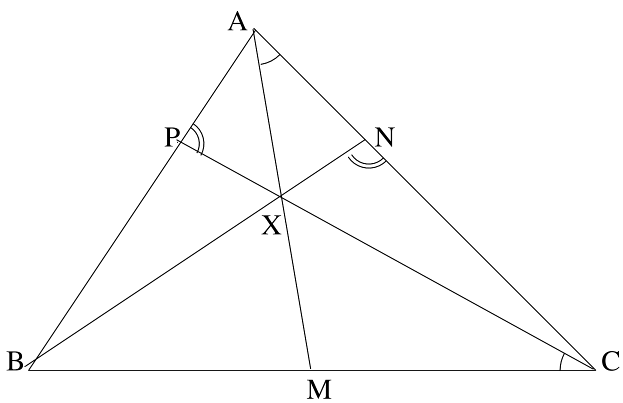
Virginia Tech Regional Math Contest 2008*

Problem 1. Find the maximum value of $xy^3 + yz^3 + zx^3 - x^3y - y^3z - z^3x$ where $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

Problem 2. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1, 3, 3, 3, 3, 3\}$ and $\{1, 3, 1, 3, 1, 3, 1, 3\}$.)

Problem 3. Find the area of the region of points (x, y) in the xy-plane such that $x^4 + y^4 \le x^2 - x^2y^2 + y^2$.

Problem 4. Let ABC be a triangle, let M be the midpoint of BC, and let X be a point on AM. Let BX meet AC at N, and let CX meet AB at P. If $\angle MAC = \angle BCP$, prove that $\angle BNC = \angle CPA$.



Problem 5. Let $a_1, a_2, \ldots$ be a sequence of nonnegative real numbers and let π, ρ be permutations of the positive integers $\mathbb{N}$ (thus $\pi, \rho : \mathbb{N} \to \mathbb{N}$ are one-to-one and onto maps).

Suppose that $\sum_{n=1}^{\infty} a_n = 1$ and ε is a real number such that

$$\sum_{n=1}^{\infty} |a_n - a_{\pi n}| + \sum_{n=1}^{\infty} |a_n - a_{\rho n}| < \varepsilon.$$

Prove that there exists a finite subset X of $\mathbb{N}$ such that $|X \cap \pi X|, |X \cap \rho X| > (1 - \varepsilon)|X|$ (here $|X|$ indicates the number of elements in X; also the inequalities $<, >$ are strict).

Problem 6. Find all pairs of positive (nonzero) integers a, b such that $ab - 1$ divides $a^4 - 3a^2 + 1$.

Problem 7. Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$ for n a positive integer. Thus $f_2(x) = x^x$ and $f_3(x) = x^{(x^x)}$. Now define $g(x) = \lim_{n \to \infty} \dfrac{1}{f_n(x)}$ for $x > 1$. Is g continuous on the open interval $(1, \infty)$? Justify your answer.

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*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`