Virginia Tech Regional Math Contest 2011*

Problem 1. Evaluate $\displaystyle\int_1^4 \frac{x-2}{(x^2+4)\sqrt{x}}dx$.

Problem 2. A sequence (a_n) is defined by $a_0 = -1$, $a_1 = 0$, and

$$a_{n+1} = a_n^2 - (n+1)^2 a_{n-1} - 1$$

for all positive integers n. Find a_{100}.

Problem 3. Find $\displaystyle\sum_{k=1}^{\infty} \frac{k^2-2}{(k+2)!}$.

Problem 4. Let m, n be positive integers and let $[a]$ denote the residue class mod mn of the integer a (thus $\{[r] \mid r \text{ is an integer}\}$ has exactly mn elements). Suppose the set $\{[ar] \mid r \text{ is an integer}\}$ has exactly m elements. Prove that there is a positive integer q such that q is prime to mn and $[nq] = [a]$.

Problem 5. Find $\displaystyle\lim_{x\to\infty} (2x)^{1+\frac{1}{2x}} - x^{1+\frac{1}{x}} - x$.

Problem 6. Let S be a set with an asymmetric relation $<$; this means that if $a, b \in S$ and $a < b$, then we do not have $b < a$. Prove that there exists a set T containing S with an asymmetric relation $\prec$ with the property that if $a, b \in S$, then $a < b$ if and only if $a \prec b$, and if $x, y \in T$ with $x \prec y$, then there exists $t \in T$ such that $x \prec t \prec y$ ($t \in T$ means "t is an element of T").

Problem 7. Let $P(x) = x^{100} + 20x^{99} + 198x^{98} + a_{97}x^{97} + \cdots + a_1 x + 1$ be a polynomial where the a_i ($1 \le i \le 97$) are real numbers. Prove that the equation $P(x) = 0$ has at least one complex root (i.e., a root of the form $a + bi$ with a, b real numbers and $b \neq 0$).

*Source: `https://personal.math.vt.edu/plinnell/Vtregional/`