

Virginia Tech Regional Math Contest 2012*

Problem 1. Evaluate

$$\int_0^{\pi/2} \frac{\cos^4 x + \sin x \cos^3 x + \sin^2 x \cos^2 x + \sin^3 x \cos x}{\sin^4 x + \cos^4 x + 2 \sin x \cos^3 x + 2 \sin^2 x \cos^2 x + 2 \sin^3 x \cos x} dx.$$

Problem 2. Solve in real numbers the equation $3x - x^3 = \sqrt{x+2}$.

Problem 3. Find nonzero complex numbers a, b, c, d, e such that

$$\begin{aligned} a + b + c + d + e &= -1 \\ a^2 + b^2 + c^2 + d^2 + e^2 &= 15 \\ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} &= -1 \\ \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} &= 15 \\ abcde &= -1 \end{aligned}$$

Problem 4. Define $f(n)$ for n a positive integer by $f(1) = 3$ and $f(n+1) = 3^{f(n)}$. What are the last two digits of $f(2012)$?

Problem 5. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{\ln n} - \left(\frac{1}{\ln n}\right)^{(n+1)/n}$ is convergent.

Problem 6. Define a sequence (a_n) for n a positive integer inductively by $a_1 = 1$ and $a_n = \frac{n}{\prod_{1 \leq d|n, d < n} a_d}$. Thus

$a_2 = 2, a_3 = 3, a_4 = 2$, etc. Find a_{999000} .

Problem 7. Let A_1, A_2, A_3 be 2×2 matrices with entries in $\mathbb{C}$ (the complex numbers). Let tr denote the trace of a matrix (so $\text{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$). Suppose $\{A_1, A_2, A_3\}$ is closed under matrix multiplication (i.e., given i, j , there exists k such that $A_i A_j = A_k$), and $\text{tr}(A_1 + A_2 + A_3) \neq 3$. Prove that there exists i such that $A_i A_j = A_j A_i$ for all j (here i, j are 1, 2 or 3).

*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>