

Virginia Tech Regional Math Contest 2014*

Problem 1. Find $\sum_{n=2}^{\infty} \frac{n^2 - 2n - 4}{n^4 + 4n^2 + 16}$.

Problem 2. Evaluate $\int_0^2 \frac{(16 - x^2)x}{16 - x^2 + \sqrt{(4 - x)(4 + x)(12 + x^2)}} dx$.

Problem 3. Find the least positive integer n such that 2^{2014} divides $19^n - 1$.

Problem 4. Suppose we are given a 19×19 chessboard (a table with 19^2 squares) and remove the central square. Is it possible to tile the remaining $19^2 - 1 = 360$ squares with 4×1 and 1×4 rectangles? (So each of the 360 squares is covered by exactly one rectangle.) Justify your answer.

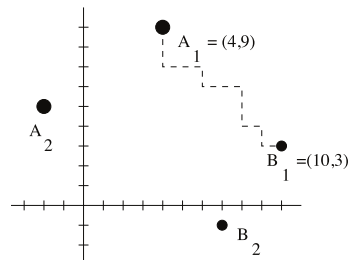
Problem 5. Let $n \geq 1$ and $r \geq 2$ be positive integers. Prove that there is no integer m such that $n(n+1)(n+2) = m^r$.

Problem 6. Let S denote the set of 2 by 2 matrices with integer entries and determinant 1 , and let T denote those matrices of S which are congruent to the identity matrix $I \pmod{3}$ (so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T$ means that $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$, and 3 divides $b, c, a - 1, d - 1$).

- (a) Let $f : T \rightarrow \mathbb{R}$ (the real numbers) be a function such that for every $X, Y \in T$ with $Y \neq I$, either $f(XY) > f(X)$ or $f(XY^{-1}) > f(X)$ (or both). Show that given two finite nonempty subsets A, B of T , there are matrices $a \in A$ and $b \in B$ such that if $a' \in A$, $b' \in B$ and $a'b' = ab$, then $a' = a$ and $b' = b$.
- (b) Show that there is no $f : S \rightarrow \mathbb{R}$ such that for every $X, Y \in S$ with $Y \neq \pm I$, either $f(XY) > f(X)$ or $f(XY^{-1}) > f(X)$.

Problem 7. Let A, B be two points in the plane with integer coordinates $A = (x_1, y_1)$ and $B = (x_2, y_2)$. (Thus $x_i, y_i \in \mathbb{Z}$, for $i = 1, 2$.) A path $\pi : A \rightarrow B$ is a sequence of **down** and **right** steps, where each step has an integer length, and the initial step starts from A , the last step ending at B . In the figure below, we indicated a path from $A_1 = (4, 9)$ to $B_1 = (10, 3)$. The distance $d(A, B)$ between A and B is the number of such paths. For example, the distance between $A = (0, 2)$ and $B = (2, 0)$ equals 6 . Consider now two pairs of points in the plane $A_i = (x_i, y_i)$ and $B_i = (u_i, z_i)$ for $i = 1, 2$, with integer coordinates, and in the configuration shown in the picture (but with arbitrary coordinates):

- $x_2 < x_1$ and $y_1 > y_2$, which means that A_1 is North-East of A_2 ; $u_2 < u_1$ and $z_1 > z_2$, which means that B_1 is North-East of B_2 .
- Each of the points A_i is North-West of the points B_j , for $1 \leq i, j \leq 2$. In terms of inequalities, this means that $x_i < \min\{u_1, u_2\}$ and $y_i > \max\{z_1, z_2\}$ for $i = 1, 2$.



*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>

- (a) Find the distance between two points A and B as before, as a function of the coordinates of A and B . Assume that A is North-West of B .
- (b) Consider the 2×2 matrix $M = \begin{pmatrix} d(A_1, B_1) & d(A_1, B_2) \\ d(A_2, B_1) & d(A_2, B_2) \end{pmatrix}$. Prove that for any configuration of points A_1, A_2, B_1, B_2 as described before, $\det M > 0$.