

Virginia Tech Regional Math Contest 2016*

Problem 1. Evaluate $\int_1^2 \frac{\ln x}{2 - 2x + x^2} dx$.

Problem 2. Determine the real numbers k such that $\sum_{n=1}^{\infty} \left(\frac{(2n)!}{4^n n! n!} \right)^k$ is convergent.

Problem 3. Let n be a positive integer and let $M_n(\mathbb{Z}_2)$ denote the n by n matrices with entries from the integers mod 2. If $n \geq 2$, prove that the number of matrices A in $M_n(\mathbb{Z}_2)$ satisfying $A^2 = 0$ (the matrix with all entries zero) is an even positive integer.

Problem 4. For a positive integer a , let $P(a)$ denote the largest prime divisor of $a^2 + 1$. Prove that there exist infinitely many triples (a, b, c) of distinct positive integers such that $P(a) = P(b) = P(c)$.

Problem 5. Suppose that m, n, r are positive integers such that

$$1 + m + n\sqrt{3} = (2 + \sqrt{3})^{2r-1}$$

Prove that m is a perfect square.

Problem 6. Let A, B, P, Q, X, Y be square matrices of the same size. Suppose that

$$\begin{array}{ll} A + B + AB = XY & AX = XQ \\ P + Q + PQ = YX & PY = YB. \end{array}$$

Prove that $AB = BA$.

Problem 7. Let q be a real number with $|q| \neq 1$ and let k be a positive integer. Define a Laurent polynomial $f_k(X)$ in the variable X , depending on q and k , by

$$f_k(X) = \prod_{i=0}^{k-1} (1 - q^i X)(1 - q^{i+1} X^{-1}).$$

Show that the constant term of $f_k(X)$, i.e., the coefficient of X^0 in $f_k(X)$, is equal to

$$\frac{(1 - q^{k+1})(1 - q^{k+2}) \cdots (1 - q^{2k})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}.$$

*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>