

Virginia Tech Regional Math Contest 2017*

Problem 1. Determine the number of real solutions to the equation $\sqrt{2-x^2} = \sqrt[3]{3-x^3}$.

Problem 2. Evaluate $\int_0^{\pi/2} \frac{dx}{1 + \cos x + \sin x}$ for $-\pi/2 < a < \pi$.

Use your answer to show that $\int_0^{\pi/2} \frac{dx}{1 + \cos x + \sin x} = \ln 2$.

Problem 3. Let ABC be a triangle and let P be a point in its interior. Suppose $\angle BAP = 10^\circ$, $\angle ABP = 20^\circ$, $\angle PCA = 30^\circ$ and $\angle PAC = 40^\circ$. Find $\angle PBC$.

Problem 4. Let P be an interior point of a triangle of area T . Through the point P , draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let a, b and c be the areas of the three triangles. Prove that $\sqrt{T} = \sqrt{a} + \sqrt{b} + \sqrt{c}$.

Problem 5. Let $f(x, y) = \frac{x+y}{2}$, $g(x, y) = \sqrt{xy}$, $h(x, y) = \frac{2xy}{x+y}$, and let

$$S = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \neq b \text{ and } f(a, b), g(a, b), h(a, b) \in \mathbb{N}\},$$

where $\mathbb{N}$ denotes the positive integers. Find the minimum of f over S .

Problem 6. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients such that $f(1) = -1$, $f(4) = 2$ and $f(8) = 34$. Suppose $n \in \mathbb{Z}$ is an integer such that $f(n) = n^2 - 4n - 18$. Determine all possible values for n .

Problem 7. Find all pairs (m, n) of nonnegative integers for which $m^2 + 2 \cdot 3^n = m(2^{n+1} - 1)$.

*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>