

Virginia Tech Regional Math Contest 2018*

Problem 1. It is known that $\int_1^2 \frac{\arctan(1+x)}{x} dx = q\pi \ln(2)$ for some rational number q . Determine q . Here $0 \leq \arctan(x) < \pi/2$ for $0 \leq x < \infty$.

Problem 2. Let $A, B \in M_6(\mathbb{Z})$ such that $A \equiv I \equiv B \pmod{3}$ and $A^3 B^3 A^3 = B^3$. Prove that $A = I$. Here $M_6(\mathbb{Z})$ indicates the 6 by 6 matrices with integer entries, I is the identity matrix, and $X \equiv Y \pmod{3}$ means all entries of $X - Y$ are divisible by 3.

Problem 3. Prove that there is no function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(f(n)) = n + 1$. Here $\mathbb{N}$ is the positive integers $\{1, 2, 3, \dots\}$.

Problem 4. Let m, n be integers such that $n \geq m \geq 1$. Prove that $\frac{\gcd(m, n)}{n} \binom{n}{m}$ is an integer. Here $\gcd$ denotes greatest common divisor and $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ denotes the binomial coefficient.

Problem 5. For $n \in \mathbb{N}$, let $a_n = \int_0^{1/\sqrt{n}} |1 + e^{it} + e^{2it} + \dots + e^{nit}| dt$.

Determine whether the sequence $(a_n) = a_1, a_2, \dots$ is bounded.

Problem 6. For $n \in \mathbb{N}$, define

$$a_n = \frac{1 + 1/3 + 1/5 + \dots + 1/(2n-1)}{n+1} \text{ and } b_n = \frac{1/2 + 1/4 + 1/6 + \dots + 1/(2n)}{n}.$$

Find the maximum and minimum of $a_n - b_n$ for $1 \leq n \leq 999$.

Problem 7. A continuous function $f : [a, b] \rightarrow [a, b]$ is called piecewise monotone if $[a, b]$ can be subdivided into finitely many subintervals

$$I_1 = [c_0, c_1], I_2 = [c_1, c_2], \dots, I_\ell = [c_{\ell-1}, c_\ell]$$

such that f restricted to each interval I_j is strictly monotone, either increasing or decreasing. Here we are assuming that $a = c_0 < c_1 < \dots < c_{\ell-1} < c_\ell = b$. We are also assuming that each I_j is a maximal interval on which f is strictly monotone. Such a maximal interval is called a lap of the function f , and the number $\ell = \ell(f)$ of distinct laps is called the lap number of f .

If $f : [a, b] \rightarrow [a, b]$ is a continuous piecewise-monotone function, show that the sequence $\left(\sqrt[n]{\ell(f^n)}\right)$ converges; here f^n means f composed with itself n -times, so $f^2(x) = f(f(x))$ etc.

*Source: <https://personal.math.vt.edu/plinnell/Vtregional/>