

# Virginia Tech Regional Math Contest 1980<sup>1</sup>

## Problem 1

Let  $*$  denote a binary operation on a set  $S$  with the property that

$$(w * x) * (y * z) = w * z \text{ for all } w, x, y, z \in S$$

Show:

- (a) If  $a * b = c$ , then  $c * c = c$ .
- (b) If  $a * b = c$ , then  $a * x = c * x$  for all  $x \in S$ .

## Problem 2

The sum of the first  $n$  terms of the sequence

$$1, (1 + 2), (1 + 2 + 2^2), \dots, (1 + 2 + \dots + 2^{k-1}), \dots$$

is of the form  $2^{n+R} + Sn^2 + Tn + U$  for all  $n > 0$ . Find  $R, S, T$  and  $U$ .

## Problem 3

Let  $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$ .

- (a) Prove that  $\lim_{n \rightarrow \infty} a_n$  exists.
- (b) Show that  $a_n = \frac{\left(1 - \left(\frac{1}{2}\right)^2\right)\left(1 - \left(\frac{1}{4}\right)^2\right)\cdots\left(1 - \left(\frac{1}{2n}\right)^2\right)}{(2n+1)a_n}$ .
- (c) Find  $\lim_{n \rightarrow \infty} a_n$  and justify your answer.

## Problem 4

Let  $P(x)$  be any polynomial of degree at most 3. It can be shown that there are numbers  $x_1$  and  $x_2$  such that  $\int_{-1}^1 P(x) dx = P(x_1) + P(x_2)$ , where  $x_1$  and  $x_2$  are independent of the polynomial  $P$ .

- (a) Show that  $x_1 = -x_2$ .
- (b) Find  $x_1$  and  $x_2$ .

## Problem 5

For  $x > 0$ , show that  $e^x < (1 + x)^{1+x}$ .

## Problem 6

Given the linear fractional transformation of  $x$  into  $f_1(x) = \frac{2x-1}{x+1}$ , define  $f_{n+1}(x) = f_1(f_n(x))$  for  $n = 1, 2, 3, \dots$ . It can be shown that  $f_{35} = f_5$ . Determine  $A, B, C$ , and  $D$  so that  $f_{28}(x) = \frac{Ax+B}{Cx+D}$ .

## Problem 7

Let  $S$  be the set of all ordered pairs of integers  $(m, n)$  satisfying  $m > 0$  and  $n < 0$ . Let  $\langle$  be a partial ordering on  $S$  defined by the statement:  $(m, n) \langle (m', n')$  if and only if  $m \leq m'$  and  $n \leq n'$ . An example is  $(5, -10) \langle (8, -2)$ . Now let  $O$  be a completely ordered subset of  $S$ , i.e., if  $(a, b) \in O$  and  $(c, d) \in O$ , then  $(a, b) \langle (c, d)$  or  $(c, d) \langle (a, b)$ . Also let  $C$  denote the collection of all such completely ordered sets.

- (a) Determine whether an arbitrary  $O \in C$  is finite.
- (b) Determine whether the cardinality  $|O|$  of  $O$  is bounded for  $O \in C$ .
- (c) Determine whether  $|O|$  can be countably infinite for any  $O \in C$ .

<sup>1</sup>VTRMC problems source: <https://personal.math.vt.edu/plinnell/Vtregional/>

### Problem 8

Let  $z = x + iy$  be a complex number with  $x$  and  $y$  rational and with  $|z| = 1$ .

- (a) Find two such complex numbers.
- (b) Show that  $|z^{2n} - 1| = 2 |\sin n\theta|$ , where  $z = e^{i\theta}$ .
- (c) Show that  $|z^{2n} - 1|$  is rational for every  $n$ .