Virginia Tech Regional Math Contest 1981[1]

Problem 1

The number $2^{48} - 1$ is exactly divisible by what two numbers between 60 and 70?

Problem 2

For which real numbers b does the function $f(x)$, defined by the conditions $f(0) = b$ and $f' = 2f - x$, satisfy $f(x) > 0$ for all $x \geq 0$?

Problem 3

Let A be non-zero square matrix with the property that $A^3 = 0$, where 0 is the zero matrix, but with A being otherwise arbitrary.

(a) Express $(I - A)^{-1}$ as a polynomial in A, where I is the identity matrix.
(b) Find a 3×3 matrix satisfying $B^2 \neq 0$, $B^3 = 0$.

end{enumerate}

Problem 4

Define $F(x)$ by $F(x) = \sum_{n=0}^{\infty} F_n x^n$ (wherever the series converges), where F_n is the nth Fibonacci number defined by $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}, n > 1$. Find an explicit closed form for $F(x)$.

Problem 5

Two elements A, B in a group G have the property $ABA^{-1}B = 1$, where 1 denotes the identity element in G.

(a) Show that $AB^2 = B^{-2}A$.
(b) Show that $AB^n = B^{-n}A$ for any integer n.
(c) Find u and v so that $\left(B^a A^b\right)\left(B^c A^d\right) = B^u A^v$.

Problem 6

With k a positive integer, prove that $\left(1 - k^{-2}\right)^k \geq 1 - \frac{1}{k}$.

Problem 7

Let $A = \{a_0, a_1, ...\}$ be a sequence of real numbers and define the sequence $A' = \{a_{0'}, a_{1'}, ...\}$ as follows for $n = 0, 1, ...$: $a_{(2n)'} = a_n$, $a_{(2n+1)'} = a_n + 1$. If $a_0 = 1$ and $A' = A$, find

(a) a_1, a_2, a_3 and a_4
(b) a_{1981}
(c) A simple general algorithm for evaluating a_n, for $n = 0, 1, ...$

Problem 8

Let

(i) $0 < a < 1$,
(ii) $0 < M_{k+1} < M_k$, for $k = 0, 1, ...$,
(iii) $\lim_{k \to \infty} M_k = 0$.

If $b_n = \sum_{k=0}^{\infty} a^{n-k} M_k$, prove that $\lim_{n \to \infty} b_n = 0$.

[1] VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/