

Virginia Tech Regional Math Contest 1982¹

Problem 1

What is the remainder when $X^{1982} + 1$ is divided by $X - 1$? Verify your answer.

Problem 2

A box contains marbles, each of which is red, white or blue. The number of blue marbles is a least half the number of white marbles and at most one-third the number of red marbles. The number which are white or blue is at least 55. Find the minimum possible number of red marbles.

Problem 3

Let a , b , and c be vectors such that $\{a, b, c\}$ is linearly dependent. Show that

$$\begin{vmatrix} a \cdot a & a \cdot b & a \cdot c \\ b \cdot a & b \cdot b & b \cdot c \\ c \cdot a & c \cdot b & c \cdot c \end{vmatrix} = 0$$

Problem 4

Prove that $t^{n-1} + t^{1-n} < t^n + t^{-n}$ when $t \neq 1$, $t > 0$ and n is a positive integer.

Problem 5

When asked to state the Maclaurin Series, a student writes (incorrectly)

$$(*) \quad f(x) = f(x) + x f'(x) + \frac{x^2}{2!} f''(x) + \frac{x^3}{3!} f'''(x) + \dots$$

- State Maclaurin's Series for $f(x)$ correctly.
- Replace the left-hand side of $(*)$ by a simple closed form expression in f in such a way that the statement becomes valid (in general).

Problem 6

Let S be a set of positive integers and let E be the operation on the set of subsets of S defined by $EA = \{x \in A \mid x \text{ is even}\}$, where $A \subseteq S$. Let CA denote the complement of A in S . $ECEA$ will denote $E(C(EA))$, etc.

- Show that $ECECEA = EA$.
- Find the maximum number of distinct subsets of S that can be generated by applying the operations E and C to a subset A of S an arbitrary number of times in any order.

Problem 7

Let $p(x) = ax^2 + bx + c$, where a , b and c are integers, with the property that $1 < p(1) < p(p(1)) < p(p(p(1)))$. Show that $a \geq 0$.

Problem 8

For $n \geq 2$, let $S_n = \sum_{k=n}^{\infty} \frac{1}{k^2}$.

- Prove or disprove that $\frac{1}{n} < S_n < \frac{1}{n-1}$.
- Prove or disprove that $S_n < \frac{1}{n-\frac{3}{4}}$.

¹VTRMC problems source: <https://personal.math.vt.edu/plinnell/Vtregional/>