Virginia Tech Regional Math Contest 1983[1]

Problem 1

In the expansion of $(a+b)^n$, where n is a natural number, there are $n+1$ dissimilar terms. Find the number of dissimilar terms in the expansion of $(a+b+c)^n$.

Problem 2

A positive integer N (in base 10) is called *special* if the operation C of replacing each digit d of N by its nine's-complement $9-d$, followed by the operation R of reversing the order of the digits, results in the original number. (For example, 3456 is a special number because $R[(C3456)] = 3456$.) Find the sum of all special positive integers less than one million which do not end in zero or nine.

Problem 3

Let a triangle have vertices at $O(0,0)$, $A(a,0)$, and $B(b,c)$ in the (x,y)-plane.

(a) Find the coordinates of a point $P(x,y)$ in the exterior of $\triangle OAB$ satisfying area$(\triangle OAP) =$ area$(\triangle OBP) =$ area$(\triangle ABP)$.
(b) Find the coordinates of a point $Q(x,y)$ in the interior of $\triangle OAB$ satisfying area$(\triangle OAQ) =$ area$(\triangle OBQ) =$ area$(\triangle ABQ)$.

Problem 4

A finite set of roads connect n towns $T_1, T_2, \ldots, T_n$ where $n \geq 2$. We say that towns T_i and T_j $(i \neq j)$ are directly connected if there is a road segment connecting T_i and T_j which does not pass through any other town. Let $f(T_k)$ be the number of other towns directly connected to T_k. Prove that f is not one-to-one.

Problem 5

Find the function $f(x)$ such that for all $L \geq 0$, the area under the graph of $y = f(x)$ and above the x -axis from $x = 0$ to $x = L$ equals the arc length of the graph from $x = 0$ to $x = L$. Hint: recall that $\frac{d}{d}x \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$.

Problem 6

Let $f(x) = \frac{1}{x}$ and $g(x) = 1 - x$ for $x \in (0,1)$. List all distinct functions that can be written in the form $f \circ g \circ f \circ g \circ \ldots \circ f \circ g \circ f$ where $\circ$ represents composition. Write each function in the form $\frac{ax+b}{cx+d}$ and prove that your list is exhaustive.

Problem 7

If a and b are real, prove that $x^4 + ax + b = 0$ cannot have only real roots.

Problem 8

A sequence f_n is generated by the recurrence formula

$$f_{n+1} = \frac{f_n f_{n-1} + 1}{f_{n-2}}$$

for $n = 2, 3, 4, \ldots$, with $f_0 = f_1 = f_2 = 1$. Prove that f_n is integer-valued for all integers $n \geq 0$.

[1]VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/