

Virginia Tech Regional Math Contest 1987¹

Problem 1

A path zig-zags from $(1, 0)$ to $(0, 0)$ along line segments $\overline{P_n P_{n+1}}$, where P_0 is $(1, 0)$ and P_n is $(2^{-n}, (-2)^{-n})$, for $n > 0$. Find the length of the path.

Problem 2

A triangle with sides of lengths a , b , and c is partitioned into two smaller triangles by the line which is perpendicular to the side of length c and passes through the vertex opposite that side. Find integers $a < b < c$ such that each of the two smaller triangles is similar to the original triangle and has sides of integer lengths.

Problem 3

Let $a_1, a_2, \dots, a_n$ be an arbitrary rearrangement of $1, 2, \dots, n$. Prove that if n is odd, then $(a_1 - 1)(a_2 - 2) \dots (a_n - n)$ is even.

Problem 4

Let $p(x)$ be given by $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ and let $|p(x)| \leq |x|$ on $[-1, 1]$.

- (a) Evaluate a_0 .
- (b) Prove that $|a_1| \leq 1$.

Problem 5

A sequence of integers $\{n_1, n_2, \dots\}$ is defined as follows: n_1 is assigned arbitrarily and, for $k > 1$,

$$n_k = \sum_{j=1}^{j=k-1} z(n_j),$$

where $z(n)$ is the number of 0s in the binary representation of n (each representation should have a leading digit of 1 except for zero which has the representation 0). An example, with $n_1 = 9$, is $\{9, 2, 3, 3, 3, \dots\}$, or in binary, $\{1001, 10, 11, 11, 11, \dots\}$.

- (a) Find n_1 so that $\lim_{k \rightarrow \infty} n_k = 31$, and calculate $n_2, n_3, \dots, n_{10}$.
- (b) Prove that, for every choice of n_1 , the sequence $\{n_k\}$ converges.

Problem 6

A sequence of polynomials is given by $p_n(x) = a_{n+2}x^2 + a_{n+1}x - a_n$, for $n \geq 0$, where $a_0 = a_1 = 1$ and, for $n \geq 0$, $a_{n+2} = a_{n+1} + a_n$. Denote by r_n and s_n the roots of $p_n(x) = 0$, with $r_n \leq s_n$. Find $\lim_{n \rightarrow \infty} r_n$ and $\lim_{n \rightarrow \infty} s_n$.

Problem 7

Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ be $n \times n$ matrices such that A^{-1} exists.

Define $A(t) = \{a_{ij}(t)\}$ and $B(t) = \{b_{ij}(t)\}$ by

- 1. $a_{ij}(t) = a_{ij}$ for $i < n$,
- 2. $a_{nj}(t) = ta_{nj}$,
- 3. $b_{ij}(t) = b_{ij}$ for $i < n$, and
- 4. $b_{nj}(t) = tb_{nj}$.

For example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $A(t) = \begin{pmatrix} 1 & 2 \\ 3t & 4t \end{pmatrix}$.

¹VTRMC problems source: <https://personal.math.vt.edu/plinnell/Vtregional/>

Prove that $A(t)^{-1}B(t) = A^{-1}B$ for $t > 0$ and any n . (Partial credit will be given for verifying the result for $n = 3$.)

Problem 8

On Halloween, a black cat and a witch encounter each other near a large mirror positioned along the y -axis. The witch is *invisible except by reflection* in the mirror. At $t = 0$, the cat is at $(10, 10)$ and the witch is at $(10, 0)$. For $t \geq 0$ the witch moves toward the cat at a speed numerically equal to their distance of separation and the cat moves toward the apparent position of the witch, as seen by reflection, at a speed numerically equal to their reflected distance of separation. Denote by $(u(t), v(t))$ the position of the cat and by $(x(t), y(t))$ the position of the witch.

- (a) Set up the equations of motion of the cat and the witch for $t \geq 0$.
- (b) Solve for $x(t)$ and $u(t)$ and find the time when the cat strikes the mirror. (Recall that the mirror is a perpendicular bisector of the line joining an object with its apparent position as seen by reflection.)