Virginia Tech Regional Math Contest 1988[1]

Problem 1

A circle C of radius r is circumscribed by a parallelogram S. Let θ denote one of the interior angles of S, with $0 < \theta \leq \frac{\pi}{2}$. Calculate the area of S as a function of r and θ.

Problem 2

A man goes into a bank to cash a check. The teller mistakenly reverses the amounts and gives the man cents for dollars and dollars for cents. (Example: if the check was for \$5.10, the man was given \$10.05.) After spending five cents, the man finds that he still has twice as much as the original check amount. What was the original check amount? Find all possible solutions.

Problem 3

Find the general solution of $y(x) + \displaystyle\int_1^x y(t)\,\mathrm{d}t = x^2$.

Problem 4

Let a be a positive integer. Find all positive integers n such that $b = a^n$ satisfies the condition that $a^2 + b^2$ is divisible by $ab + 1$.

Problem 5

Let f be differentiable on $[0, 1]$ and let $f(\alpha) = 0$ and $f(x_0) = -.0001$ for some α and $x_0 \in (0, 1)$. Also let $|f'(x)| \geq 2$ on $[0, 1]$. Find the smallest upper bound on $|\alpha - x_0|$ for all such functions.

Problem 6

Find positive real numbers a and b such that $f(x) = ax - bx^3$ has four extrema on $[-1, 1]$, at each of which $|f(x)| = 1$.

Problem 7

For any set S of real numbers define a new set $f(S)$ by $f(S) = \left\{ \frac{x}{3} \text{ divides } x \in S \right\} \cup \left\{ \frac{x+2}{3} \mid x \in S \right\}$.

(a) Sketch, carefully, the set $f(f(f(I)))$, where I is the interval $[0, 1]$.
(b) If T is a bounded set such that $f(T) = T$, determine, *with proof*, whether T can contain $\frac{1}{2}$.

Problem 8

Let $T(n)$ be the number of incongruent triangles with integral sides and perimeter $n \geq 6$. Prove that $T(n) = T(n - 3)$ if n is even, or disprove by a counterexample. (Note: two triangles are *congruent* if there is a one-to-one correspondence between the sides of the two triangles such that corresponding sides have the same length.)

[1]VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/