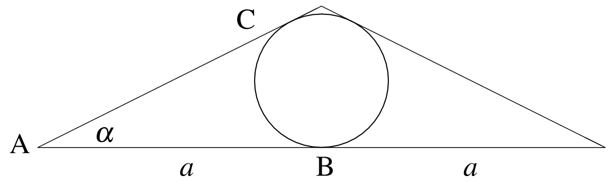
Virginia Tech Regional Math Contest 1991[1]

Problem 1

An isosceles triangle with an inscribed circle is labeled as shown in the figure. Find an expression, in terms of the angle α and the length a, for the area of the curvilinear triangle bounded by sides AB and AC and the arc BC.



Problem 2

Find all differentiable functions f which satisfy $f(x)^3 = \int_0^x f(t)^2 \, dt$ for all real x.

Problem 3

Prove that if α is a real root of $(1 - x^2)(1 + x + x^2 + \ldots + x^n) - x = 0$ which lies in $(0, 1)$, with $n = 1, 2, \ldots$, then α is also a root of $(1 - x^2)(1 + x + x^2 + \ldots + x^{n+1}) - 1 = 0$.

Problem 4

Prove that if $x > 0$ and $n > 0$, where x is real and n is an integer, then

$$\frac{x^n}{(x+1)^{n+1}} \le \frac{n^n}{(n+1)^{n+1}}.$$

Problem 5

Let $f(x) = x^5 - 5x^3 + 4x$. In each part (i)–(iv), prove or disprove that there exists a real number c for which $f(x) - c = 0$ has a root of multiplicity (i) one, (ii) two, (iii) three, (iv) four.

Problem 6

Let $a_0 = 1$ and for $n > 0$, let a_n be defined by

$$a_n = -\sum_{k=1}^{n} \frac{a_{n-k}}{k!}.$$

Prove that $a_n = \frac{(-1)^n}{n!}$, for $n = 0, 1, 2, \ldots$.

Problem 7

A and B play the following money game, where a_n and b_n denote the amount of holdings of A and B, respectively, after the nth round. At each round a player pays one-half his holdings to the bank, then receives one dollar from the bank if the **other** player had less than c dollars at the end of the **previous** round. If $a_0 = .5$ and $b_0 = 0$, describe the behavior of a_n and b_n when n is large, for

(a) $c = 1.24$ and
(b) $c = 1.26$.

Problem 8

Mathematical National Park has a collection of trails. There are designated campsites along the trails, including a campsite at each intersection of trails. The rangers call each stretch of trail

[1]VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/

between adjacent campsites a “segment”. The trails have been laid out so that it is possible to take a hike that starts at any campsite, covers each segment exactly once, and ends at the beginning campsite. Prove that it is possible to plan a collection $\mathcal{C}$ of hikes with all of the following properties:

(a) Each segment is covered exactly once in one hike $h \in \mathcal{C}$ and never in any of the other hikes of $\mathcal{C}$.
(b) Each $h \in \mathcal{C}$ has a base campsite that is its beginning and end, but which is never passed in the middle of the hike. (Different hikes of $\mathcal{C}$ may have different base campsites.)
(c) Except for its base campsite at beginning and end, no hike in $\mathcal{C}$ passes any campsite more than once.