Virginia Tech Regional Math Contest 1993[1]

Problem 1

Prove that $\int_0^1 \int_{x^2}^1 e^{y^{\frac{3}{2}}} \, \mathrm{d}y \, \mathrm{d}x = \frac{2e-2}{3}$.

Problem 2

Prove that if $f : \mathbb{R}to\mathbb{R}$ is continuous and $f(x) = \int_0^x f(t) \, \mathrm{d}t$, then $f(x)$ is identically zero.

Problem 3

Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_{n(x)}}$, for $n = 1, 2....$ Prove that $f_{n'}(1) = 1$ and $f_{n''}(1) = 2$, for all $n \geq 2$.

Problem 4

Prove that a triangle in the plane whose vertices have integer coordinates cannot be equilateral.

Problem 5

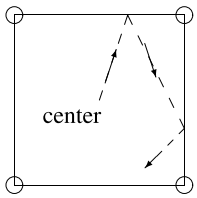
Find $\sum_{n=1}^{\infty} \frac{3^{-n}}{n}$.

Problem 6

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a surjective map with the property that if the points A, B and C are collinear, then so are $f(A)$, $f(B)$ and $f(C)$. Prove that f is bijective.

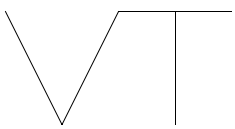
Problem 7

On a small square billiard table with sides of length 2 ft., a ball is played from the center and after rebounding off the sides several times, goes into a cup at one of the corners. Prove that the total distance travelled by the ball is bold(not) an integer number of feet.

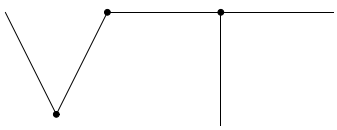


Problem 8

A popular Virginia Tech logo looks something like



Suppose that wire-frame copies of this logo are constructed of 5 equal pieces of wire welded at three places as shown:



[1]VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/

If bending is allowed, but no re-welding, show clearly how to cut the maximum possible number of ready-made copies of such a logo from the piece of welded wire mesh shown. Also, prove that no larger number is possible.