Virginia Tech Regional Math Contest 1994[1]

Problem 1

Evaluate $\int_0^1 \int_0^x \int_0^{1-x^2} e^{(1-z)^2} \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x$.

Problem 2

Let f be continuous real function, strictly increasing in an interval $[0, a]$ such that $f(0) = 0$. Let g be the inverse of f, i.e., $g(f(x)) = x$ for all x in $[0, a]$. Show that for $0 le x le a$, $0 le y le f(a)$, we have xy le integral*{0}^{x} f(t) dt + integral*{0}^{y} g(t) dt.

Problem 3

Find all continuously differentiable solutions $f(x)$ for

$$f(x)^2 = \int_0^x \left(f(t)^2 - f(t)^4 + (f'(t))^2\right) \mathrm{d}t + 100$$

where $f(0)^2 = 100$.

Problem 4

Consider the polynomial equation $ax^4 + bx^3 + x^2 + bx + a = 0$, where a and b are real numbers, and $a > \frac{1}{2}$. Find the maximum possible value of $a + b$ for which there is at least one positive real root of the above equation.

Problem 5

Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be a function which satisfies $f(0, 0) = 1$ and

$$f(m, n) + f(m + 1, n) + f(m, n + 1) + f(m + 1, n + 1) = 0$$

for all $m, n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and all real numbers, respectively). Prove that $|f(m, n)| \geq \frac{1}{3}$, for infinitely many pairs of integers (m, n).

Problem 6

Let A be an $n \times n$ matrix and let α be an n-dimensional vector such that $A\alpha = \alpha$. Suppose that all the entries of A and α are positive real numbers. Prove that α is the only linearly independent eigenvector of A corresponding to the eigenvalue 1. Hint: if β is another eigenvector, consider the minimum of $\frac{\alpha_i}{|\beta_i|}$, $i = 1, \ldots, n$, where the α_i's and β_i's are the components of α and β, respectively.

Problem 7

Define $f(1) = 1$ and $f(n + 1) = 2\sqrt{f(n)^2 + n}$ for $n \geq 1$. If $N \geq 1$ is an integer, find $\sum_{n=1}^{N} \frac{1}{f(n)^2}$.

Problem 8

Let a sequence $\{x_n\}_{n=0}^{\infty}$ of rational numbers be defined by $x_0 = 10$, $x_1 = 29$ and $x_{n+2} = \frac{19x_{n+1}}{94x_n}$ for $n \geq 0$. Find $\sum_{n=0}^{\infty} \frac{x_{6n}}{2^n}$.

[1]VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/