Virginia Tech Regional Math Contest 1996[1]

Problem 1

Evaluate $\int_0^1 \int_{\sqrt{y-y^2}}^{\sqrt{1-y^2}} xe^{(x^2+y^2)^2}\,\mathrm{d}x\,\mathrm{d}y$.

Problem 2

For each rational number r, define $f(r)$ to be the smallest positive integer n such that $r = \frac{m}{n}$ for some integer m, and denote by $P(r)$ the point in the (x, y) plane with coordinates $P(r) = \left(r, \frac{1}{f(r)}\right)$. Find a necessary and sufficient condition that, given two rational numbers r_1 and r_2 such that $0 < r_1 < r_2 < 1$,

$$P\left(\frac{r_1 f(r_1) + r_2 f(r_2)}{f(r_1) + f(r_2)}\right)$$

will be the point of intersection of the line joining $(r_1, 0)$ and $P(r_2)$ with the line joining $P(r_1)$ and $(r_2, 0)$.

Problem 3

Solve the differential equation $y^2 = e^{\frac{\mathrm{d}y}{\mathrm{d}x}}$ with the initial condition $y = e$ when $x = 1$.

Problem 4

Let $f(x)$ be a twice continuously differentiable in the interval $(0, \infty)$. If

$$\lim_{x\to\infty} \left(x^2 f''(x) + 4xf'(x) + 2f(x)\right) = 1,$$

find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} xf'(x)$. Do bold(not) assume any special form of $f(x)$. Hint: use l'Hôpital's rule.

Problem 5

Let a_i, $i = 1, 2, 3, 4$, be real numbers such that $a_1 + a_2 + a_3 + a_4 = 0$. Show that for arbitrary real numbers b_i, $i = 1, 2, 3$, the equation

$$a_1 + b_1 x + 3a_2 x^2 + b_2 x^3 + 5a_3 x^4 + b_3 x^5 + 7a_4 x^6 = 0$$

has at least one real root which is on the interval $-1 \le x \le 1$.

Problem 6

There are $2n$ balls in the plane such that no three balls are on the same line and such that no two balls touch each other. n balls are red and the other n balls are green. Show that there is at least one way to draw n line segments by connecting each ball to a unique different colored ball so that no two line segments intersect.

Problem 7

Let us define

$$\begin{aligned} f_{n,0}(x) &= x + \frac{\sqrt{x}}{n} && \text{for} \quad x > 0, n \ge 1, \\ f_{n,j+1}(x) &= f_{n,0}\left(f_{n,j}(x)\right), && j = 0, 1, \ldots, n-1. \end{aligned}$$

Find $\lim_{n\to\infty} f_{n,n}(x)$ for $x > 0$.

[1]VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/