

Virginia Tech Regional Math Contest 1999¹

Problem 1

Let G be the set of all continuous functions $f : \mathbb{R} \rightarrow \mathbb{R}$, satisfying the following properties:

- (a) $f(x) = f(x + 1)$ for all x ,
- (b) $\int_{\{0\}}^{\{1\}} f(x) \, dx = 1999$.

Show that there is a number α such that $\alpha = \int_0^1 \int_0^x f(x + y) \, dy \, dx$ for all $f \in G$.

Problem 2

Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is infinitely differentiable and satisfies both of the following properties:

- (a) $f(1) = 2$,
- (b) If α, β are real numbers satisfying $\alpha^2 + \beta^2 = 1$, then $f(\alpha x)f(\beta x) = f(x)$ for all x .

Find $f(x)$. Guesswork will not be accepted.

Problem 3

Let ε, M be positive real numbers, and let $A_1, A_2, \dots$ be a sequence of matrices such that for all n ,

- (a) A_n is an $n \times n$ matrix with integer entries,
- (b) The sum of the absolute values of the entries in each row of A_n is at most M .

If δ is a positive real number, let $e_{n(\delta)}$ denote the number of nonzero eigenvalues of A_n which have absolute value less than δ . (Some eigenvalues can be complex numbers.) Prove that one can choose $\delta > 0$ so that $\frac{e_{n(\delta)}}{n} < \varepsilon$ for all n .

Problem 4

A rectangular box has sides of length 3, 4, 5. Find the volume of the region consisting of all points that are within distance 1 of at least one of the sides.

Problem 5

Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function from the set of positive real numbers to the same set satisfying $f(f(x)) = x$ for all positive x . Suppose that f is infinitely differentiable for all positive x , and that $f(a) \neq a$ for some positive a . Prove that $\lim_{x \rightarrow \infty} f(x) = 0$.

Problem 6

A set S of distinct positive integers has property bold(N D) if no element x of S divides the sum of the integers in any subset of $S \setminus \{x\}$. Here $S \setminus \{x\}$ means the set that remains after x is removed from S .

- (a) Find the smallest positive integer n such that $\{3, 4, n\}$ has property bold(N D).
- (b) If n is the number found in (i), prove that no set S with property bold(N D) has $\{3, 4, n\}$ as a proper subset.

¹VTRMC problems source: <https://personal.math.vt.edu/plinnell/Vtregional/>