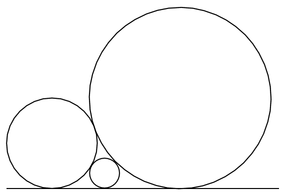
Virginia Tech Regional Math Contest 2001[1]

Problem 1

Three infinitely long circular cylinders each with unit radius have their axes along the x, y and z-axes. Determine the volume of the region common to all three cylinders. (Thus one needs the volume common to $\{y^2 + z^2 \leq 1\}$, $\{z^2 + x^2 \leq 1\}$, $\{x^2 + y^2 \leq 1\}$.)

Problem 2

Two circles with radii 1 and 2 are placed so that they are tangent to each other and a straight line. A third circle is nestled between them so that it is tangent to the first two circles and the line. Find the radius of the third circle.



Problem 3

For each positive integer n, let S_n denote the total number of squares in an $n \times n$ square grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chess board (i.e., determine S_8).

Problem 4

Let a_n be the nth positive integer k such that the greatest integer not exceeding $\sqrt{k}$ divides k, so the first few terms of $\{a_n\}$ are $\{1, 2, 3, 4, 6, 8, 9, 12, ...\}$. Find a_{10000} and give reasons to substantiate your answer.

Problem 5

Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$.

That is, determine the real numbers x for which the above power series converges; you must determine correctly whether the series is convergent at the end points of the interval.

Problem 6

Find a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(f(x)) = \frac{3x+1}{x+3}$ for all positive real numbers x (here $\mathbb{R}^+$ denotes the positive (nonzero) real numbers).

Problem 7

Let G denote a set of invertible 2×2 matrices (matrices with complex numbers as entries and determinant nonzero) with the property that if a, b are in G, then so are ab and a^{-1}. Suppose there exists a function $f : G \to \mathbb{R}$ with the property that either $f(ga) > f(a)$ or $f(g^{-1}a) > f(a)$ for all a, g in G with $g \neq I$ (here I denotes the identity matrix, $\mathbb{R}$ denotes the real numbers, and the inequality signs are strict inequality). Prove that given finite nonempty subsets A, B of G, there is a matrix in G which can be written in exactly one way in the form xy with x in A and y in B.

[1] VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/