Virginia Tech Regional Math Contest 2007[1]

Problem 1

Evaluate $\displaystyle\int_0^x \frac{\mathrm{d}\theta}{2+\tan(\theta)}$, where $0 \leq x \leq \frac{\pi}{2}$.

Use your result to show that $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{2+\tan(\theta)} = \frac{\pi+\ln\left(\frac{9}{8}\right)}{10}$.

Problem 2

Given that $e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$ find, in terms of e, the exact values of

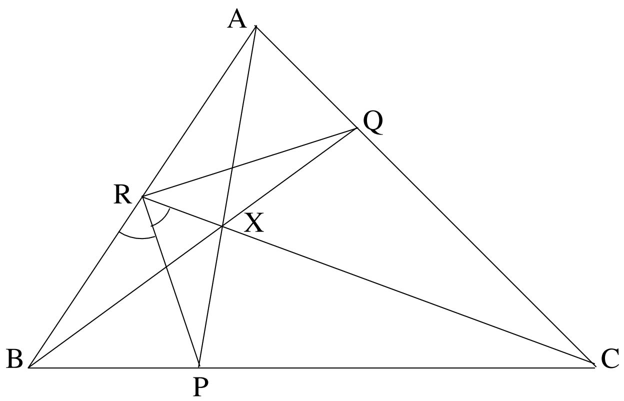
(a) $\frac{1}{1!} + \frac{2}{3!} + \frac{3}{5!} + \ldots + \frac{n}{(2n-1)!} + \ldots$ and
(b) $\frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \ldots + \frac{n}{(2n+1)!} + \ldots$

Problem 3

Solve the initial value problem $\frac{\mathrm{d}y}{\mathrm{d}x} = y \ln y + y e^x$, $y(0) = 1$ (i.e., find y in terms of x).

Problem 4

In the diagram below, P, Q, R are points on BC, CA, AB respectively such that the lines AP, BQ, CR are concurrent at X. Also PR bisects $\angle BRC$, i.e. $\angle BRP = \angle PRC$. Prove that $\angle PRQ = 90°$.



Problem 5

Find the third digit after the decimal point of

$$\left(2+\sqrt{5}\right)^{100}\left(\left(1+\sqrt{2}\right)^{100}+\left(1+\sqrt{2}\right)^{-100}\right).$$

For example, the third digit after the decimal point of $\pi = 3.14159\ldots$ is 1.

Problem 6

Let n be a positive integer, let A, B be square symmetric $n \times n$ matrices with real entries (so if a_{ij} are the entries of A, the a_{ij} are real numbers and $a_{ij} = a_{ji}$). Suppose there are $n \times n$ matrices X, Y (with complex entries) such that $\det(AX + BY) \neq 0$. Prove that $\det(A^2 + B^2) \neq 0$ (det indicates the determinant).

[1]VTRMC problems source: https://personal.math.vt.edu/plinnell/Vtregional/

Problem 7

Determine whether the series $\sum_{n=2}^{\infty} n^{-(1+(\ln(\ln n))^{-2})}$ is convergent or divergent (ln denotes natural log).