

Virginia Tech Regional Math Contest 2009¹

Problem 1

A walker and a jogger travel along the same straight line in the same direction. The walker walks at one meter per second, while the jogger runs at two meters per second. The jogger starts one meter in front of the walker. A dog starts with the walker, and then runs back and forth between the walker and the jogger with constant speed of three meters per second. Let $f(n)$ meters denote the total distance travelled by the dog when it has returned to the walker for the n th time (so $f(0) = 0$). Find a formula for $f(n)$.

Problem 2

Given that

$$40! = abc \text{ def } 283247897734345611269596115894272 \text{ pqr stu vwx}$$

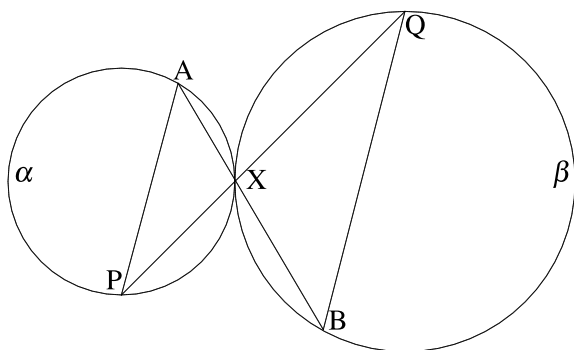
find $p, q, r, s, t, u, v, w, x$, and then find a, b, c, d, e, f .

Problem 3

Define $f(x) = \int_0^x \int_0^x e^{u^2 v^2} du dv$. Calculate $2f''(2) + f'(2)$ (here $f'(x) = \frac{d}{dx} f(x)$).

Problem 4

Two circles α, β touch externally at the point X . Let A, P be two distinct points on α different from X , and let AX and PX meet β again in the points B and Q respectively. Prove that AP is parallel to QB .



Problem 5

Let $\mathbb{C}$ denote the complex numbers and let $M_3(\mathbb{C})$ denote the 3 by 3 matrices with entries in $\mathbb{C}$. Suppose $A, B \in M_3(\mathbb{C})$, $B \neq 0$, and $AB = 0$ (where 0 denotes the 3 by 3 matrix with all entries zero). Prove that there exists $0 \neq D \in M_3(\mathbb{C})$ such that $AD = DA = 0$.

Problem 6

Let n be a nonzero integer. Prove that $n^4 - 7n^2 + 1$ can never be a perfect square (i.e., of the form m^2 for some integer m).

Problem 7

Does there exist a twice differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f'(x) = f(x+1) - f(x)$ for all x and $f''(0) \neq 0$? Justify your answer. (Here $\mathbb{R}$ denotes the real numbers and f' denotes the derivative of f .)

¹VTRMC problems source: <https://personal.math.vt.edu/plinnell/Vtregional/>