

Virginia Tech Regional Math Contest 2013¹

Problem 1

Let $I = 3\sqrt{2} \int_0^x \frac{\sqrt{1 + \cos t}}{17 - 8 \cos t} dt$. If $0 < x < \pi$ and $\tan I = \frac{2}{\sqrt{3}}$, what is x ?

Problem 2

Let ABC be a right-angled triangle with $\angle ABC = 90^\circ$, and let D be on AB such that $AD = 2DB$. What is the maximum possible value of $\angle ACD$?

Problem 3

Define a sequence (a_n) for $n \geq 1$ by $a_1 = 2$ and $a_{n+1} = a_n^{1+n^{-\frac{3}{2}}}$. Is (a_n) convergent (i.e., $\lim_{n \rightarrow \infty} a_n < \infty$)?

Problem 4

A positive integer n is called **special** if it can be represented in the form $n = \frac{x^2 + y^2}{u^2 + v^2}$, for some positive integers x, y, u, v . Prove that

- (a) 25 is special;
- (b) 2013 is not special;
- (c) 2014 is not special.

Problem 5

Prove that $\frac{x}{\sqrt{1+x^2}} + \frac{y}{\sqrt{1+y^2}} + \frac{z}{\sqrt{1+z^2}} \leq \frac{3\sqrt{3}}{2a}$ for any positive real numbers x, y, z such that $x + y + z = xyz$.

Problem 6

Let $X = \begin{pmatrix} 7 & 8 & 9 \\ 8 & -9 & -7 \\ -7 & -7 & 9 \end{pmatrix}$, $Y = \begin{pmatrix} 9 & 8 & -9 \\ 8 & -7 & 7 \\ 7 & 9 & 8 \end{pmatrix}$, let $A = Y^{-1} - X$ and let B be the inverse of $X + A^{-1}$.

Find a matrix M such that $M^2 = XY - BY$ (you may assume that A and $X^{-1} + A^{-1}$ are invertible).

Problem 7

Find $\sum_{n=1}^{\infty} \frac{n}{(2^n + 2^{-n})^2} + \frac{(-1)^n n}{(2^n - 2^{-n})^2}$.

¹VTRMC problems source: <https://personal.math.vt.edu/plinnell/Vtregional/>