

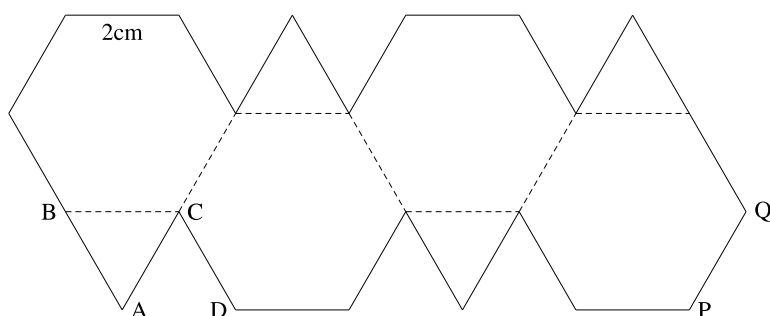
Virginia Tech Regional Math Contest 2015¹

Problem 1

Find all integers n for which $n^4 + 6n^3 + 11n^2 + 3n + 31$ is a perfect square.

Problem 2

The planar diagram below, with equilateral triangles and regular hexagons, sides length 2cm, is folded along the dashed edges of the polygons, to create a closed surface in three-dimensional Euclidean space. Edges on the periphery of the planar diagram are identified (or glued) with precisely one other edge on the periphery in a natural way. Thus for example, BA will be joined to QP and AC will be joined to DC . Find the volume of the three-dimensional region enclosed by the resulting surface.



Problem 3

Let $(a_i)_{(1 \leq i \leq 2015)}$ be a sequence consisting of 2015 integers, and let $(k_i)_{(1 \leq i \leq 2015)}$ be a sequence of 2015 positive integers (positive integer excludes 0). Let

$$A = \begin{pmatrix} a_{k_1}^1 & a_{k_1}^2 & \dots & a_{k_1}^{2015} \\ a_{k_2}^1 & a_{k_2}^2 & \dots & a_{k_2}^{2015} \\ v \dots & v \dots & d \dots & v \dots \\ a_{k_{2015}}^1 & a_{k_{2015}}^2 & \dots & a_{k_{2015}}^{2015} \end{pmatrix}$$

Prove that $2015!$ divides $\det A$.

Problem 4

Consider the harmonic series $\sum_{n \geq 1} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} \dots$. Prove that every positive rational number can be obtained as an **unordered** partial sum of this series. (An unordered partial sum may skip some of the terms $\frac{1}{k}$.)

Problem 5

Evaluate $\int_0^\infty \frac{\arctan(\pi x) - \arctan(x)}{x} dx$ (where $0 \leq \arctan(x) < \frac{\pi}{2}$ for $0 \leq x < \infty$).

Problem 6

Let $(a_1, b_1), \dots, (a_n, b_n)$ be n points in $\mathbb{R}^2$ (where $\mathbb{R}$ denotes the real numbers), and let $\varepsilon > 0$ be a positive number. Can we find a real-valued function $f(x, y)$ that satisfies the following three conditions?

- (a) $f(0, 0) = 1$;

¹VTRMC problems source: <https://personal.math.vt.edu/plinnell/Vtregional/>

- (b) $f_n(x, y) \neq 0$ for only finitely many $(x, y) \in \mathbb{R}^2$;
 (c) $\sum_{r=1}^n |f(x + a_r, y + b_r) - f(x, y)| < \varepsilon$ for every $(x, y) \in \mathbb{R}^2$.

Justify your answer.

Problem 7

Let n be a positive integer and let $x_1, \dots, x_n$ be n nonzero points in $\mathbb{R}^2$. Suppose $\langle x_i, x_j \rangle$ (scalar or dot product) is a rational number for all i, j ($1 \leq i, j \leq n$). Let S denote all points of $\mathbb{R}^2$ of the form $\sum_{i=1}^n a_i x_i$ where the a_i are integers. A closed disk of radius R and center P is the set of points at distance at most R from P (includes the points distance R from P). Prove that there exists a positive number R and closed disks $D_1, D_2, \dots$ of radius R such that

- (a) Each disk contains exactly two points of S ;
 (b) Every point of S lies in at least one disk;
 (c) Two distinct disks intersect in at most one point.